

4.1 Exponential Functions and Their Graphs

In this section you will learn to:

- evaluate exponential functions
- graph exponential functions
- use transformations to graph exponential functions
- use compound interest formulas

An **exponential function** f with base b is defined by

$$f(x) = b^x \text{ or } y = b^x, \text{ where } b > 0, b \neq 1, \text{ and } x \text{ is any real number.}$$

Note: Any transformation of $y = b^x$ is also an exponential function.

Example 1: Determine which functions are exponential functions. For those that are not, explain why they are not exponential functions.

(a) $f(x) = 2^x + 7$ Yes No _____

(b) $g(x) = x^2$ Yes No _____

(c) $h(x) = 1^x$ Yes No _____

(d) $f(x) = x^x$ Yes No _____

(e) $h(x) = 3 \cdot 10^{-x}$ Yes No _____

(f) $f(x) = -3^{x+1} + 5$ Yes No _____

(g) $g(x) = (-3)^{x+1} + 5$ Yes No _____

(h) $h(x) = 2x - 1$ Yes No _____

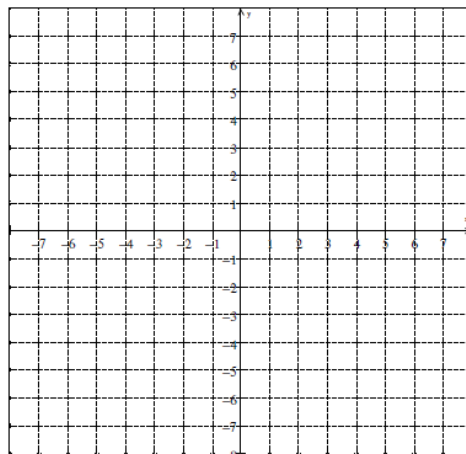
Example 2: Graph each of the following and find the domain and range for each function.

(a) $f(x) = 2^x$ domain: _____

range: _____

(b) $g(x) = \left(\frac{1}{2}\right)^x$ domain: _____

range: _____



Characteristics of Exponential Functions $f(x) = b^x$	
$b > 1$	$0 < b < 1$
Domain:	
Range:	

Transformations of $g(x) = b^x$ ($c > 0$): (Order of transformations is H S R V.)

Horizontal: $g(x) = b^{x+c}$ (graph moves c units left)
 $g(x) = b^{x-c}$ (graph moves c units right)

Stretch/Shrink: $g(x) = cb^x$ (graph stretches if $c > 1$)
 (Vertical) (graph shrinks if $0 < c < 1$)

Stretch/Shrink: $g(x) = b^{cx}$ (graph shrinks if $c > 1$)
 (Horizontal) (graph stretches if $0 < c < 1$)

Reflection: $g(x) = -b^x$ (graph reflects over the x -axis)
 $g(x) = b^{-x}$ (graph reflects over the y -axis)

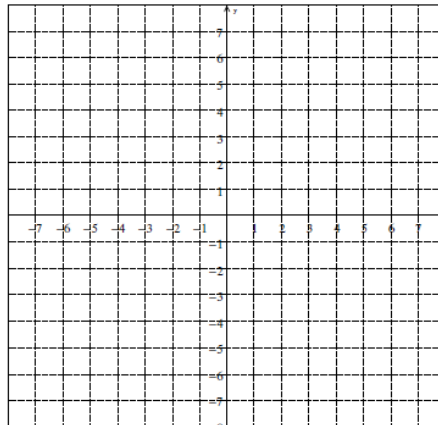
Vertical: $g(x) = b^x + c$ (graph moves up c units)
 $g(x) = b^x - c$ (graph moves down c units)

Example 3: Use $f(x) = 2^x$ to obtain the graph $g(x) = -2^{x+3} - 1$.

Domain of g : _____

Range of g : _____

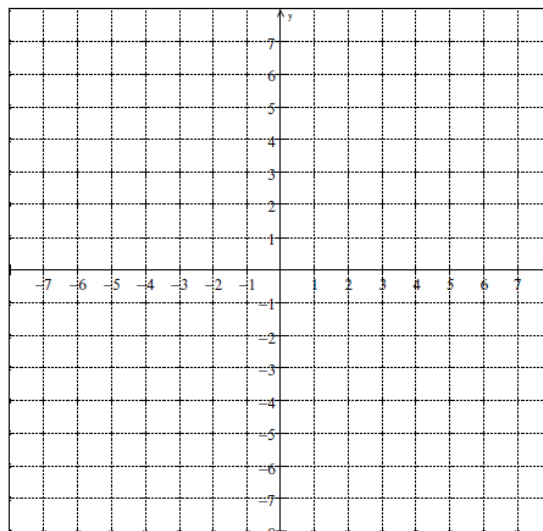
Equation of any asymptote(s) of g : _____



$f(x) = e^x$ is called the **natural exponential function**,
 where the irrational number e (approximately 2.718282) is called the **natural base**.

(The number e is defined as the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as n gets larger and larger.)

Example 4: Graph $f(x) = e^x$, $g(x) = e^{x-3}$, and $h(x) = -e^x$ on the same set of axes.



Vertical:

$g(x) = b^x + c$ (graph moves up c units)

$g(x) = b^x - c$ (graph moves down c units)

Periodic Interest Formula	Continuous Interest Formula
$A = P \left(1 + \frac{r}{n} \right)^{nt}$	$A = Pe^{rt}$
<p> A = balance in the account (<u>A</u>mount after t years) P = <u>p</u>rincipal (beginning amount in the account) r = annual interest <u>r</u>ate (as a decimal) n = <u>n</u>umber of times interest is compounded per year t = time (in years) </p>	

Example 5: Find the accumulated value of a \$5000 investment which is invested for 8 years at an interest rate of 12% compounded:

- (a) annually

- (b) semi-annually

- (c) quarterly

- (d) monthly

- (e) continuously

4.1 Homework Problems

1. Use a calculator to find each value to four decimal places.

(a) $5^{\sqrt{3}}$ (b) 7^{π} (c) $2^{-5.3}$ (d) e^2 (e) e^{-2} (f) $-e^{0.25}$ (g) π^{-1}

2. Simplify each expression without using a calculator. (Recall: $b^n \cdot b^m = b^{n+m}$ and $(b^m)^n = b^{mn}$)

(a) $6^{\sqrt{2}}6^{\sqrt{2}}$ (b) $(3^{\sqrt{2}})^{\sqrt{2}}$ (c) $(b^{\sqrt{2}})^{\sqrt{8}}$ (d) $(5^{\sqrt{3}})^{\sqrt{3}}$ (e) $4^{\frac{1}{2}}4^{\frac{1}{2}}$ (f) $b^{\sqrt{12}}b^{\sqrt{3}}$

For Problems 3 – 14, graph each exponential function. State the domain and range for each along with the equation of any asymptotes. Check your graph using a graphing calculator.

3. $f(x) = 3^x$ 4. $f(x) = -(3^x)$ 5. $f(x) = 3^{-x}$ 6. $f(x) = \left(\frac{1}{3}\right)^x$

7. $f(x) = 2^x - 3$ 8. $f(x) = 2^{x-3}$ 9. $f(x) = 2^{x+5} - 5$ 10. $f(x) = -2^{-x}$

11. $f(x) = -2^{x+3} + 1$ 12. $f(x) = \left(\frac{1}{2}\right)^{x-3} - 4$ 13. $f(x) = e^{-x} + 2$ 14. $f(x) = -e^{x+2}$

15. \$10,000 is invested for 5 years at an interest rate of 5.5%. Find the accumulated value if the money is (a) compounded semiannually; (b) compounded quarterly; (c) compounded monthly; (d) compounded continuously.

16. Sam won \$150,000 in the Michigan lottery and decides to invest the money for retirement in 20 years. Find the accumulated value for Sam's retirement for each of his options:

- (a) a certificate of deposit paying 5.4% compounded yearly
- (b) a money market certificate paying 5.35% compounded semiannually
- (c) a bank account paying 5.25% compounded quarterly
- (d) a bond issue paying 5.2% compounded daily
- (e) a saving account paying 5.19% compounded continuously

4.1 Homework Answers: 1. (a) 16.2425; (b) 451.8079; (c) .0254; (d) 7.3891; (e) .1353; (f) -1.2840; (g) .3183 2. (a) $36^{\sqrt{2}}$; (b) 9; (c) b^4 ; (d) 125; (e) 4; (f) $b^{3\sqrt{3}}$ 3. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$ 4. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$ 5. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$ 6. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$ 7. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$; $y = -3$ 8. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$ 9. Domain: $(-\infty, \infty)$; Range: $(-5, \infty)$; $y = -5$ 10. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$ 11. Domain: $(-\infty, \infty)$; Range: $(-\infty, 1)$; $y = 1$ 12. Domain: $(-\infty, \infty)$; Range: $(-4, \infty)$; $y = -4$ 13. Domain: $(-\infty, \infty)$; Range: $(2, \infty)$; $y = 2$ 14. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$ 15. (a) \$13,116.51; (b) \$13,140.67; (c) \$13,157.04; (d) \$13,165.31 16. (a) \$429,440.97; (b) \$431,200.96; (c) \$425,729.59; (d) \$424,351.12; (e) \$423,534.64

4.2 Applications of Exponential Functions

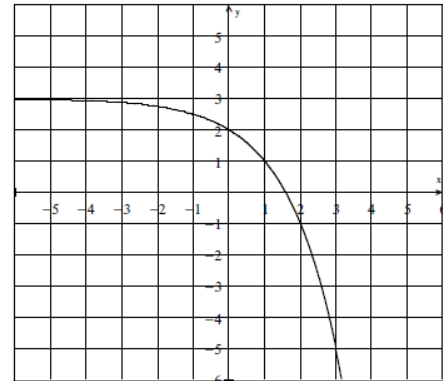
In this section you will learn to:

- find exponential equations using graphs
- solve exponential growth and decay problems
- use logistic growth models

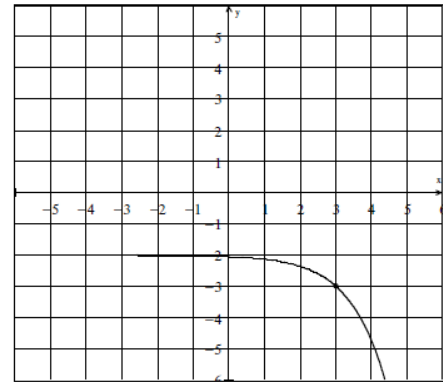
Example 1: The graph of g is the transformation of $f(x) = 2^x$. Find the equation of the graph of g .

HINTS:

1. There are no stretches or shrinks.
2. Look at the general graph and asymptote to determine any reflections and/or vertical shifts.
3. Follow the point $(0, 1)$ on f through the transformations to help determine any vertical and/or horizontal shifts.



Example 2: The graph of g is the transformation of $f(x) = e^x$. Find the equation of the graph of g .



Example 3: In 1969, the world population was approximately 3.6 billion, with a growth rate of 1.7% per year. The function $f(x) = 3.6e^{0.017x}$ describes the world population, $f(x)$, in billions, x years after 1969. Use this function to estimate the world population in

1969 _____ 2000 _____ 2012 _____

Example 4: The exponential function $f(x) = 84.5(1.012)^x$ models the population of Mexico, $f(x)$, in millions, x years after 1986.

- (a) Without using a calculator, substitute 0 for x and find Mexico's population in 1986.
- (b) Estimate Mexico's population, to the nearest million in the year 2000.
- (c) Estimate Mexico's population, to the nearest million, this year.

Example 5: College students study a large volume of information. Unfortunately, people do not retain information for very long. The function $f(x) = 80e^{-0.5x} + 20$ describes the percentage of information, $f(x)$, that a particular person remembers x weeks after learning the information (without repetition).

- (a) Substitute 0 for x and find the percentage of information remembered at the moment it is first learned.
- (b) What percentage of information is retained after 1 week? _____ 4 weeks? _____ 1 year? _____

Radioactive Decay Formula:

The amount A of radioactive material present at time t is given by $A = A_0(2)^{\frac{t}{h}}$ where A_0 is the amount that was present initially (at $t = 0$) and h is the material's half-life.

Example 6: The half-life of radioactive carbon-14 is 5700 years. How much of an initial sample will remain after 3000 years?

Example 7: The half-life of Arsenic-74 is 17.5 days. If 4 grams of Arsenic-74 are present in a body initially, how many grams are presents 90 days later?

Logistic Growth Models: Logistic growth models situations when there are factors that limit the ability to grow or spread. From population growth to the spread of disease, nothing on earth can exhibit exponential growth indefinitely. Eventually this growth levels off and approaches a maximum level (which can be represented by a horizontal asymptote).

Logistic growth models are used in the study of conservation biology, learning curves, spread of an epidemic or disease, carrying capacity, etc. The mathematical model for limited logistic growth is given

by: $f(t) = \frac{c}{1 + ae^{-bt}}$ or $A = \frac{c}{1 + ae^{-bt}}$, where a , b , and c are constants, $c > 0$ and $b > 0$.

As time increases ($t \rightarrow \infty$), the expression $ae^{-bt} \rightarrow \underline{\hspace{2cm}}$ and $A \rightarrow \underline{\hspace{2cm}}$.

Therefore $y = c$ is a **horizontal asymptote** for the graph of the function. Thus c represents the limiting size.

Example 8: The function $f(t) = \frac{200,000}{1 + 1999e^{-0.06t}}$ describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a town with 200,000 inhabitants.

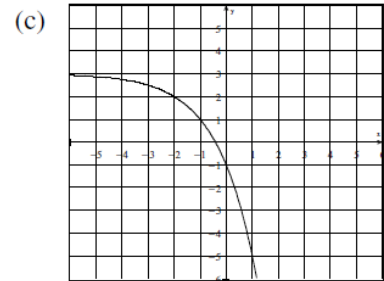
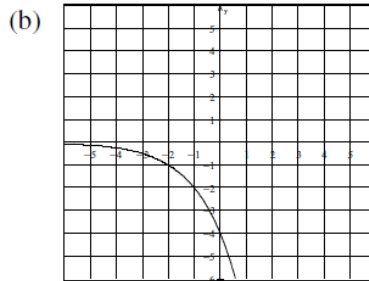
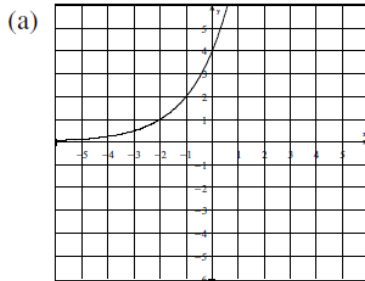
- (a) How many people became ill with the flu when the epidemic began? $\underline{\hspace{2cm}}$
- (b) How many people were ill by the end of the 4th week? $\underline{\hspace{2cm}}$
- (c) What is the limiting size of $f(t)$, the population that becomes ill? $\underline{\hspace{2cm}}$
- (d) What is the horizontal asymptote for this function? $\underline{\hspace{2cm}}$

Example 9: The function $f(t) = \frac{0.8}{1 + e^{-0.2t}}$ is a model for describing the proportion of correct responses, $f(t)$, after t learning trials.

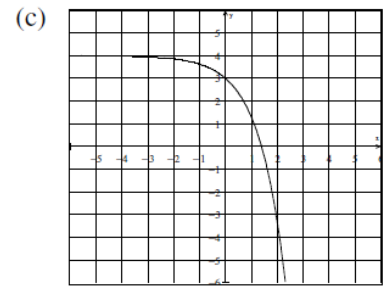
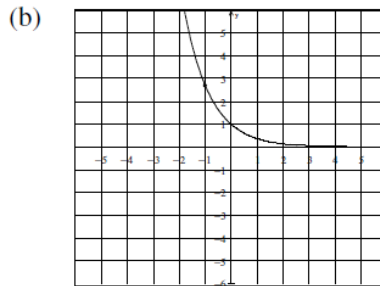
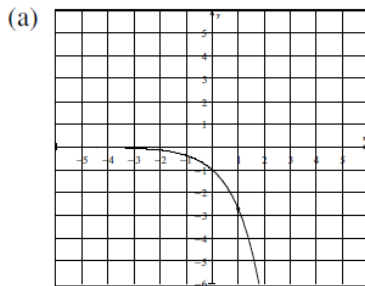
- (a) Find the proportion of correct responses prior to learning trials taking place. $\underline{\hspace{2cm}}$
- (b) Find the proportion of correct responses after 10 learning trials. $\underline{\hspace{2cm}}$
- (c) What is the limiting size of $f(t)$ as continued trials take place? $\underline{\hspace{2cm}}$
- (d) What is the horizontal asymptote for this function? $\underline{\hspace{2cm}}$
- (e) Sketch a graph of this function.

4.2 Homework Problems

1. Find the equation of each exponential function, $g(x)$, whose graph is shown. Each graph involves one or more transformation of the graph of $f(x) = 2^x$.



2. Find the equation of each exponential function, $g(x)$, whose graph is shown. Each graph involves one or more transformation of the graph of $f(x) = e^x$.



3. In 1970, the U. S. population was approximately 203.3 million, with a growth rate of 1.1% per year. The function $f(x) = 203.3e^{0.011x}$ describes the U. S. population, $f(x)$, in millions, x years after 1970. Use this function to estimate the U. S. population in the year 2012.
4. A common bacterium with an initial population of 1000 triples every day. This is modeled by the formula $P(t) = 1000(3)^t$, where $P(t)$ is the total population after t days. Find the total population after
- (a) 12 hours (b) 1 day (c) 1½ days (d) 2 days
5. Assuming the rate of inflation is 5% per year, the predicted price of an item can be modeled by the function $P(t) = P_0(1.05)^t$, where P_0 represents the initial price of the item and t is in years.
- (a) Based on this information, what will the price of a new car be in the year 2012, if it cost \$20,000 in the year 2000?
- (b) Estimate the price of a gallon of milk be in the year 2012, if it cost \$2.95 in the year 2000? Round your estimate to the nearest cent.

6. The 1986 explosion at the Chernobyl nuclear power plant in the former Soviet Union sent about 1000 kilograms of radioactive cesium-137 into the atmosphere. The function $f(x) = 1000(0.5)^{\frac{x}{30}}$ describes the amount, $f(x)$, in kilograms, of cesium-137 remaining in Chernobyl x years after 1986. If even 100 kilograms of cesium-137 remain in Chernobyl's atmosphere, the area is considered unsafe for human habitation. Find $f(60)$ and determine if Chernobyl will be safe for human habitation by 2046.
7. The logistic growth function $f(t) = \frac{100,000}{1 + 5000e^{-t}}$ describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a particular community.
- How many people became ill with the flu when the epidemic began?
 - How many people were ill by the end of the fifth week?
 - What is the limiting size of the population that becomes ill?
8. The logistic growth function $P(x) = \frac{90}{1 + 271e^{-0.122x}}$ models the percentage, $P(x)$ of Americans who are x years old with some coronary heart disease.
- What percentage of 20-year-olds have some coronary heart disease?
 - What percentage of 80-year-olds have some coronary heart disease?

4.2 Homework Answers: 1. (a) $g(x) = 2^{x+2}$; (b) $g(x) = -2^{x+2}$; (c) $g(x) = -2^{x+2} + 3$ 2. (a) $g(x) = -e^x$;
(b) $g(x) = e^{-x}$; (c) $g(x) = 4 - e^x$ 3. about 322.7 million 4. (a) about 1732; (b) 3000;
(c) about 5196; (d) 9000 5. (a) about \$35,917.13; (b) \$5.30 6. 250; no 7. (a) about 20 people;
(b) about 2883 people; (c) 100,000 people 8. (a) about 3.7%; (b) about 88.6%

4.3 Logarithmic Functions and Their Graphs

In this section you will learn to:

- change logarithmic form \leftrightarrow exponential form
- evaluate natural and common logarithms
- use basic logarithmic properties
- graph logarithmic functions
- use transformations to graph logarithmic functions

The **logarithmic function** with base b is the function $f(x) = \log_b x$. For $x > 0$ and $b > 0, b \neq 1$,

$$y = \log_b x \text{ is equivalent to } b^y = x.$$

Example 1: Complete the table below:

Logarithmic Form	Exponential Form	Answer
$\log_{10} 100 = x$		
$3 = \log_7 x$		
$2 = \log_b 25$		
$\log_2 8 = a$		
$\log_{10} 10$		
$\log_e e = x$		
$\log_{27} 3$		
	$6^x = \frac{1}{36}$	
	$b^0 = 1$	
	$2^3 = x$	
	$e^1 = x$	
	$b^2 = 36$	

Example 2: Evaluate each of the following logarithms “mentally” without using a calculator:

$$\log_{10} 100 = \underline{\hspace{2cm}} \quad \log_{10} 1000 = \underline{\hspace{2cm}} \quad \log_5 125 = \underline{\hspace{2cm}} \quad \log_5 25 = \underline{\hspace{2cm}}$$

$$\log_5 5 = \underline{\hspace{2cm}} \quad \log_5 1 = \underline{\hspace{2cm}} \quad \log_5 \left(\frac{1}{25} \right) = \underline{\hspace{2cm}} \quad \log_{10} .01 = \underline{\hspace{2cm}}$$

$$\log_{25} 5 = \underline{\hspace{2cm}} \quad \log_8 2 = \underline{\hspace{2cm}} \quad \log_4 2 = \underline{\hspace{2cm}} \quad \log_2 2^{-5} = \underline{\hspace{2cm}}$$

$$\log_7 7 = \underline{\hspace{2cm}} \quad \log_3 3 = \underline{\hspace{2cm}} \quad \log_7 1 = \underline{\hspace{2cm}} \quad \log_3 1 = \underline{\hspace{2cm}}$$

Basic Logarithmic Properties Involving One	$\log_b b = 1$ because $\log_b 1 = 0$ because
Inverse Properties	$\log_b b^x = x$ because $b^{\log_b x} = x$ because

Example 3: Evaluate each of the following without a calculator.

$$\log_{10} 10^5 = \underline{\hspace{2cm}} \quad \log_e e^4 = \underline{\hspace{2cm}} \quad 10^{\log_{10} 9} = \underline{\hspace{2cm}}$$

$$e^{\log_e 3} = \underline{\hspace{2cm}} \quad 5^{\log_5 n} = \underline{\hspace{2cm}} \quad \log_2 \frac{1}{2^n} = \underline{\hspace{2cm}}$$

$$5^{\log_5 2x} = \underline{\hspace{2cm}} \quad \log_2 2^{x+2} = \underline{\hspace{2cm}} \quad 10^{\log \sqrt[3]{x}} = \underline{\hspace{2cm}}$$

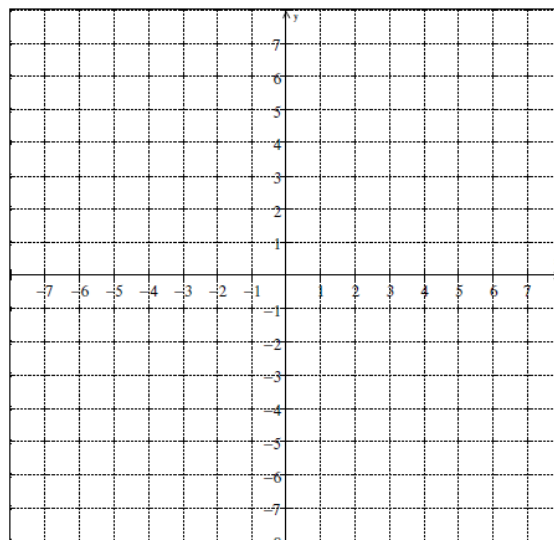
$f(x) = b^x$ and $f^{-1}(x) = \log_b x$ are **inverse functions** of each other.

If $f(x) = 2^x$, then $f^{-1} =$ _____

If $f(x) = 10^x$, then $f^{-1} =$ _____

If $f(x) = e^x$, then $f^{-1} =$ _____

Example 4: Graph $f(x) = 2^x$ and $g(x) = \log_2 x$.



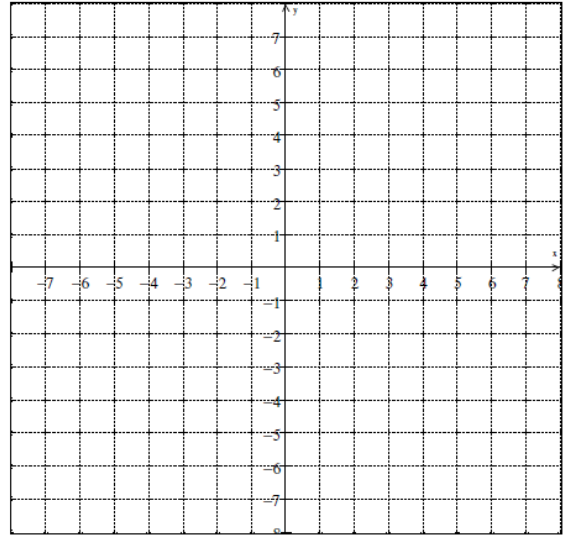
Characteristics of Inverse Functions:			
$y = b^x$		$y = \log_b x$	
Domain:	Range:	Domain:	Range:

Example 5: Use $f(x) = \log_2 x$ to obtain the graph $g(x) = \log_2(x+3) + 4$. Also find the domain, range, and the equation of any asymptotes of g .

Domain: _____

Range: _____

Asymptote(s): _____

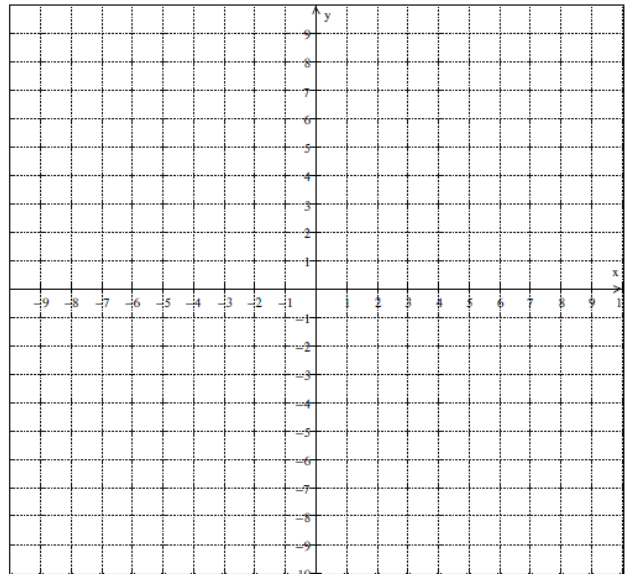


Example 6: Use $f(x) = \log_3 x$ to obtain the graph $g(x) = \log_3(4-x)$. Also find the domain, range, and the equation of any asymptotes.

Domain: _____

Range: _____

Asymptote(s): _____



Common Logs		
Natural Logs		

Logarithmic Properties		
General Logarithm (base = b)	Common Logarithm (base = 10)	Natural Logarithm (base = e)

Example 7: Evaluate each of the following without a calculator:

$$\log 10^{4.2} = \underline{\hspace{2cm}} \quad 10^{\log a^2} = \underline{\hspace{2cm}} \quad \ln e^3 = \underline{\hspace{2cm}} \quad e^{\ln(x+2)} = \underline{\hspace{2cm}}$$

$$\log_3(\log_2 8) = \underline{\hspace{2cm}} \quad \log_3(\log_3(\log_2 8)) = \underline{\hspace{2cm}} \quad \ln(\log 10) = \underline{\hspace{2cm}}$$

Example 8: Solve each of the equations by changing to exponential form.

$$\log_3(x-5) = 2$$

$$\log_5 x^2 = 2$$

$$\log_2(x^2 - 2x) = 3$$

Example 9: Let $f(x) = 2^{x+5} - 16$

- (a) Find the domain and range of f .
- (b) Find the equation of the asymptote for the graph of f .
- (c) Evaluate $f(-3)$.
- (d) Find the x and y -intercepts of f .
- (e) Find an equation for the inverse of f .
- (f) Find the domain and range of the inverse.

Example 10: Let $f(x) = \log_3(x+9) - 4$

- (a) Find the domain and range of f .
- (b) Find the equation of the asymptote for the graph of f .
- (c) Evaluate $f(18)$.
- (d) Find the x and y -intercepts of f .
- (e) Find an equation for the inverse of f .
- (f) Find the domain and range of the inverse.

4.3 Homework Problems

Write each equation in logarithmic form.

1. $5^3 = 125$ 2. $3^{-2} = \frac{1}{9}$ 3. $\sqrt{49} = 7$ 4. $m^n = p$ 5. $\left(\frac{1}{3}\right)^{-3} = 27$

Write each equation in exponential form.

6. $\log_4 64 = 3$ 7. $\log_{10} 1000 = 3$ 8. $\log_\pi \pi = 1$ 9. $\log_5 1 = 0$ 10. $\log_2 \frac{1}{8} = -3$

Find each value of x without using a calculator.

11. $\log_8 64 = x$ 12. $\log_{64} 8 = x$ 13. $\log_2 8 = x$ 14. $\log_8 2 = x$ 15. $\log_{\frac{1}{2}} \frac{1}{8} = x$

16. $\log_{\frac{1}{2}} 8 = x$ 17. $\log_9 x = 2$ 18. $\log_5 x = 1$ 19. $\log_x 7 = 1$ 20. $\log_8 x = 0$

21. $3^{\log_3 5} = x$ 22. $\pi^{\log_\pi x} = 7$ 23. $\log_3 \sqrt{3} = x$ 24. $\log_9 \frac{1}{9} = x$ 25. $x^{\log_8 2} = 2$

26. $\ln e^2 = x$ 27. $\ln x = 1$ 28. $\ln x = 0$ 29. $\ln(x-2) = 0$ 30. $\ln x = -1$

31. Graph $f(x) = 2^x + 1$ and $g(x) = \log_2(x-1)$ on the same graph. Find the domain and range of each and then determine whether f and g are inverse functions.

For problems 32 - 35, use the graph of $f(x) = \log_3 x$ and transformations of f to find the domain, range, and asymptotes of g .

32. $g(x) = \log_3(x+3)$ 33. $g(x) = 3 + \log_3 x$ 34. $g(x) = -\log_3(-x)$ 35. $g(x) = -\frac{1}{2} \log_3(x-5)$

For problems 36 - 39, use the graph of $f(x) = \ln x$ and transformations of f to find the domain, range, and asymptotes of g .

36. $g(x) = 3 \ln x$ 37. $g(x) = \ln 3x$ 38. $g(x) = 5 + \ln(x+2)$ 39. $g(x) = \ln(5-x)$

Use a calculator to find each value to four decimal places.

40. $\log_{10} 17$ 41. $\log 3.5$ 42. $\log_e 5$ 43. $\ln 63$ 44. $(\log 2)(\ln 2)$

Evaluate or simplify each expression without using a calculator.

45. $\log 1000$ 46. $\log \frac{1}{1000}$ 47. $\log \sqrt{10}$ 48. $10^{\log 7}$ 49. $\log 1$

50. $\ln 1$ 51. $\ln e$ 52. $\ln e^7$ 53. $\ln \frac{1}{e^7}$ 54. $\ln e^x$

55. $\log_3(\log_2 8)$ 56. $\log_3(\log_3(\log_3 27))$ 57. $\ln(\log_4(\log_2 16))$ 58. $\log(\ln e)$

Solve each of the logarithmic equations by first changing the equation into exponential form.

59. $\log_5(x-5) = 3$ 60. $\log_3 2x = 2$ 61. $\log_2(x+1) = -2$ 62. $\log_4(x^2 - 3x) = 1$

Find the inverse function, $f^{-1}(x)$, for each function.

63. $f(x) = \frac{1}{3} \ln x$ 64. $f(x) = \log(x-3)$ 65. $f(x) = e^{4x}$ 66. $f(x) = 5^{x+3}$

For each of the functions below, find (a) the domain and range, (b) the equation of the asymptote of the graph, (c) the x - and y -intercepts, (d) the equation for the inverse function, and (e) the domain and range of the inverse function.

67. $f(x) = 2^{x+3} - 4$

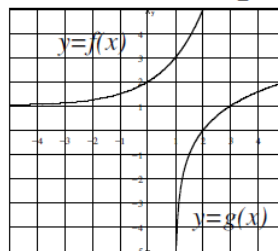
68. $f(x) = \log_3(x+9) - 4$

4.3 Homework Answers: 1. $\log_5 125 = 3$ 2. $\log_3\left(\frac{1}{9}\right) = -2$ 3. $\log_{49} 7 = \frac{1}{2}$ 4. $\log_m p = n$

5. $\log_{\frac{1}{3}} 27 = -3$ 6. $4^3 = 64$ 7. $10^3 = 1000$ 8. $\pi^1 = \pi$ 9. $5^0 = 1$ 10. $2^{-3} = \frac{1}{8}$ 11. 2 12. $\frac{1}{2}$

13. 3 14. $\frac{1}{3}$ 15. 3 16. -3 17. 81 18. 5 19. 7 20. 1 21. 5 22. 7 23. $\frac{1}{2}$ 24. -1 25. 8

26. $x = 2$ 27. $x = e$ 28. $x = 1$ 29. $x = 3$ 30. $x = \frac{1}{e}$



31. Domain of f : $(-\infty, \infty)$; Range of f : $(1, \infty)$; Domain of g : $(1, \infty)$;

Range of g : $(-\infty, \infty)$; f and g are inverse functions

32. Domain: $(-3, \infty)$; Range: $(-\infty, \infty)$; Asymptote: $x = -3$

33. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Asymptote: $x = 0$

34. Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$; Asymptote: $x = 0$ 35. Domain: $(5, \infty)$; Range: $(-\infty, \infty)$;

Asymptote: $x = 5$ 36. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Asymptote: $x = 0$ 37. Domain: $(0, \infty)$;

Range: $(-\infty, \infty)$; Asymptote: $x = 0$ 38. Domain: $(-2, \infty)$; Range: $(-\infty, \infty)$; Asymptote: $x = -2$

39. Domain: $(-\infty, 5)$; Range: $(-\infty, \infty)$; Asymptote: $x = 5$ 40. 1.2304 41. .5441 42. 1.6094

43. 4.1431 44. .2087 45. 3 46. -3 47. $\frac{1}{2}$ 48. 7 49. 0 50. 0 51. 1 52. 7 53. -7

54. x 55. 1 56. 0 57. 0 58. 0 59. $\{130\}$ 60. $\{9/2\}$ 61. $\{-3/4\}$ 62. $\{-1, 4\}$ 63. $f^{-1}(x) = e^{3x}$

64. $f^{-1}(x) = 10^x + 3$ 65. $f^{-1}(x) = \frac{1}{4} \ln x$ 66. $f^{-1}(x) = \log_5 x - 3$ 67. (a) Domain: $(-\infty, \infty)$;

Range: $(-4, \infty)$; (b) $y = -4$; (c) x -int: -1; y -int: 4; (d) $f^{-1}(x) = \log_2(x+4) - 3$; (e) Domain: $(-4, \infty)$;

Range: $(-\infty, \infty)$ 68. (a) Domain: $(-9, \infty)$; Range: $(-\infty, \infty)$; (b) $x = -9$; (c) x -int: 72; y -int: -2;

(d) $f^{-1}(x) = 3^{x+4} - 9$; (e) Domain: $(-\infty, \infty)$; Range: $(-9, \infty)$

4.4 Applications of Logarithmic Functions

In this section you will learn to:

- use logarithms to solve geology problems
- use logarithms to solve charging battery problems
- use logarithms to solve population growth problems

Richter Scale	If R is the intensity of an earthquake, A is the amplitude (measured in micrometers), and P is the period of time (the time of one oscillation of the Earth's surface, measured in seconds), then $R = \log \frac{A}{P}$
Charging Batteries	If M is the theoretical maximum charge that a battery can hold and k is a positive constant that depends on the battery and the charger, the length of time t (in minutes) required to charge the battery to a given level C is given by $t = -\frac{1}{k} \ln \left(1 - \frac{C}{M} \right)$
Population Doubling Time	If r is the annual growth rate and t is the time (in years) required for a population to double, then $t = \frac{\ln 2}{r}$

Example 1: Find the intensity of an earthquake with amplitude of 4000 micrometers and a period of 0.07 second.

Example 2: An earthquake has a period of $\frac{1}{4}$ second and an amplitude of 6 cm. Find its measure on the Richter scale. (Hint: 1 cm = 10,000 micrometers.)

Example 3: How long will it take to bring a fully discharged battery to 80% of full charge? Assume that $k = 0.025$ and that time is measured in minutes.

Example 4: The population of the Earth is growing at the approximate rate of 1.7% per year. If this rate continues, how long will it take the population to double?

4.5 Properties of Logarithms

In this section you will learn to:

- use the product, quotient, and power rules
- expand and condense logarithmic expressions
- use the change-of-base property

Properties of Exponents	$b^{-n} = \frac{1}{b^n}$ $b^0 = 1$ $b^n \cdot b^m = b^{n+m}$ $(b^m)^n = b^{mn}$ $\frac{b^m}{b^n} = b^{m-n}$ $(ab)^n = a^n b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
Logarithmic Properties Involving One	$\log_b b = 1$ $\log_b 1 = 0$
Inverse Properties	$\log_b b^x = x$ $b^{\log_b x} = x$

Product Rule	$\log_b (MN) = \log_b M + \log_b N$	
Quotient Rule	$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$	
Power Rule	$\log_b M^p = p \log_b M$	
<i>M, N, and b are positive real numbers with $b \neq 1$.</i>		

Example 1: Use the **product rule** to expand the logarithmic expressions. $\log_b (MN) = \log_b M + \log_b N$

(a) $\log 3x$

(b) $\log 1000x$

(c) $\log 1000 =$

(d) $\ln (x^2 + 2x)$

(e) $\ln ex$

(f) $\ln 3xy(z + 1)$

Example 2: Use the **quotient rule** to expand the logarithmic expressions. $\log_b \left(\frac{M}{N} \right) = \log_b M - \log_b N$

(a) $\log \left(\frac{x}{2} \right)$

(b) $\ln \left(\frac{3e}{2} \right)$

(c) $\ln \left(\frac{A}{BC} \right)$

(d) $\log_5 \left(\frac{25}{x} \right)$

(e) $\ln \left(\frac{e^5}{3} \right)$

(f) $\log_2 \left(\frac{8}{5y} \right)$

Example 3: Use the **power rule** to expand the logarithmic expressions. $\log_b M^p = p \log_b M$

(a) $\log x^2$

(b) $\log x^3$

(c) $\ln \sqrt{x}$

(d) $\ln \sqrt[3]{x}$

(e) $\log_5 \sqrt[3]{x^2}$

Beware of these frequently occurring log errors!!!

Example 4: Use properties of logarithms to expand each logarithmic expression as much as possible. Simplify whenever possible.

(a) $\log_b (x\sqrt{y})$

(b) $\log_5 \left(\frac{\sqrt[3]{a}}{25b^4} \right)$

(c) $\ln \left(\frac{e^3 x^3 \sqrt{x+2}}{3y^2} \right)$

Example 5: Let $\log_b 2 = A$, $\log_b 3 = B$, and $\log_b 5 = C$. Write each expression below in terms of A , B , and C .

(a) $\log_b 6$

(b) $\log_b \frac{9}{10}$

(c) $\log_b \frac{3}{8\sqrt{5}}$

Condensing Logarithmic Expressions (Write as a single logarithm with a coefficient of 1.)		
Product Rule	Quotient Rule	Power Rule
$\log_b M + \log_b N = \log_b (MN)$	$\log_b M - \log_b N = \log_b \left(\frac{M}{N} \right)$	$p \log_b M = \log_b M^p$

Example 6: Use properties of logarithms to condense each logarithmic expression as much as possible. Write the expression as a single logarithm with a coefficient of 1. Simplify whenever possible.

(a) $\log 4 + \log 25$

(b) $\log_4 2 + \log_4 8$

(c) $\ln 3 + \ln x - \ln 5$

(d) $3 \log x - \log(x + 2)$

(e) $2 \ln x + \ln(x + 1)$

(f) $6 \ln x + \frac{1}{2} \ln y - 2 \ln x - \ln 2$

$$(g) \frac{1}{2}(\ln a - \ln bc) + 5 \ln(d + e)$$

$$(h) 2 \log x - \log(x^2 - 25) + \log 10 + \log(x - 5)$$

Change of Base Property: $\log_b M = \frac{\log_a M}{\log_a b} = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}$

Example 7: Evaluate each of the following using your calculator. Round to 4 decimal places:

(a) $\log_2 8 =$ _____ (b) $\log_5 13 =$ _____ (c) $\log_{13} 5 =$ _____ (d) $\log_x 100 =$ _____

Example 8: Write each of the following as a single term that does not contain any logarithm:

(a) $e^{\ln 10x^5 - \ln 2x}$

(b) $3^{\frac{1}{2} \log_3 x - 4 \log_3 x}$

(c) $10^{3 \log xy + 2 \log y - \log x}$

4.5 Homework Problems

Determine whether each statement is true or false.

1. $\log_b ab = \log_b a + 1$
2. $\log_b \frac{1}{a} = -\log_b a$
3. $\log_b 0 = 1$
4. $\log_b 2 = \log_2 b$
5. $\log_b 2 = \frac{1}{\log_2 b}$
6. $\log_b (x + y) = \log_b x + \log_b y$
7. $\ln x + \ln 2x = \ln 3x$
8. $\ln \sqrt{x} = \frac{\ln x}{2}$
9. $\ln(8x^3) = 3\ln(2x)$
10. $x \log 10^x = x^2$
11. $\frac{\log A}{\log B} = \log A - \log B$
12. $\ln(5x) + \ln 1 = \ln(5x)$

Use properties of logarithms to expand each expression. Simplify whenever possible.

13. $\log 9x$
14. $\log \frac{100}{x}$
15. $\log_2 \frac{x}{16}$
16. $\ln \frac{e^3}{5}$
17. $\log_b bx^5$
18. $\log xy^{-5}$
19. $\log_4 \frac{x\sqrt{y}}{16}$
20. $\log \frac{x^2}{yz^3}$
21. $\ln \sqrt{ex}$
22. $\log_b \frac{ab^5}{c^2d}$
23. $\ln \frac{e^2}{b\sqrt{e}}$
24. $\log_6 \frac{\sqrt[3]{x}}{36y^4}$

Let $\log_b 2 = A$, $\log_b 3 = B$, and $\log_b 5 = C$. Write each logarithmic expression in terms of A , B , and C .

25. $\log_b 10$
26. $\log_b \frac{5}{4}$
27. $\log_b \frac{\sqrt{5}}{6}$
28. $\log_b \frac{27}{20}$

Use properties of logarithms to condense each logarithmic expression as much as possible. Write the expression as a single logarithm with a coefficient of 1. Simplify whenever possible.

29. $\log_3 3 + \log_3 9$
30. $\log 8 + \log 125$
31. $\log_2 96 - \log_2 3$
32. $\ln 4 + 2\ln x - \ln 9$
33. $\ln e - \ln 1 + 3\ln x$
34. $-2\log x + \log xy$
35. $5\log x - \frac{1}{3}\log y$
36. $\ln(x^2 - 1) - \ln(x + 1)$

Evaluate each of the following using your calculator. Round answers to four decimal places.

37. $\log_2 3$
38. $\log_3 2$
39. $\log_5 \pi$
40. $\log_\pi 5$
41. $\log_{\frac{1}{3}} 2$
42. $\log_{2.5} 2$

Write each of the following as a single term without logarithms.

43. $e^{\ln x + \ln y}$
44. $10^{2\log x - 3\log x}$
45. $e^{\ln 6x^3 - 2\ln 3x}$
46. $3^{\frac{1}{2}\log_3 y - \log_3 x}$
47. $10^{6\log xy + \log y - 3\log x}$

4.5 Homework Answers: 1. True 2. True 3. False 4. False 5. True 6. False 7. False 8. True
9. True 10. True 11. False 12. True 13. $\log 9 + \log x$ 14. $2 - \log x$ 15. $\log_2 x - 4$ 16. $3 - \ln 5$
17. $1 + 5 \log_b x$ 18. $\log x - 5 \log y$ 19. $\log_4 x + \frac{1}{2} \log_4 y - 2$ 20. $2 \log x - \log y - 3 \log z$
21. $\frac{1}{2} + \frac{1}{2} \ln x$ 22. $\log_b a + 5 - 2 \log_b c - \log_b d$ 23. $\frac{3}{2} - \ln b$ 24. $\frac{1}{3} \log_6 x - 2 - 4 \log_6 y$ 25. $A + C$
26. $C - 2A$ 27. $\frac{1}{2} C - A - B$ 28. $3B - 2A - C$ 29. 3 30. 3 31. 5 32. $\ln \frac{4x^2}{9}$ 33. $1 + \ln x^3$
34. $\log \frac{y}{x}$ 35. $\log \frac{x^5}{\sqrt[3]{y}}$ 36. $\ln(x-1)$ 37. 1.5850 38. .6309 39. .7113 40. 1.4060 41. -.6309
42. -0.5 43. xy 44. $\frac{1}{x}$ 45. $\frac{2x}{3}$ 46. $\frac{\sqrt{y}}{x}$ 47. $x^3 y^7$

4.6 Exponential and Logarithmic Equations (Part I)

In this section you will learn to:

- solve exponential equations using like bases
- solve exponential equations using logarithms
- solve logarithmic equations using the definition of a logarithm
- solve logarithmic equations using 1-to-1 properties of logarithms
- apply logarithmic and exponential equations to real-world problems
- convert $y = ab^x$ to an exponential equation using base e

Definition of a Logarithm	$y = \log_b x$ is equivalent to $b^y = x$
Inverse Properties	$\log_b b^x = x$ $b^{\log_b x} = x$
Log Properties Involving One	$\log_b b = 1$ $\log_b 1 = 0$
Product Rule	$\log_b (MN) = \log_b M + \log_b N$
Quotient Rule	$\log_b \left(\frac{M}{N} \right) = \log_b M - \log_b N$
Power Rule	$\log_b M^p = p \log_b M$
One-to-One Properties	If $b^M = b^N$ then $M = N$.
	If $\log_b M = \log_b N$ then $M = N$.
	If $M = N$ then $\log_b M = \log_b N$.

Example 1: Solve each equation by expressing each side as a power of the same base.

(a) $5^{x+1} = 25^{x-3}$

(b) $9^{2x} = \frac{1}{\sqrt[5]{3}}$

(c) $e^2 e^x = \frac{e^6}{e^x}$

Example 2: Solve $3^x = 30$ using the one-to-one property for (a) common logarithms and (b) natural logarithms.

Example 3: Solve $5e^{2x} = 60$

Steps for solving exponential equations:

1.

2.

3.

4.

Example 4: Solve $10^x + 3 = 835$

Example 5: Solve $5^{x-2} = 50$

Example 6: Solve $2^{x+2} = 3^{x-1}$

Example 7: Use factoring to solve each of the following equations. (Hint: Use substitution or short-cut method learned in Section 1.6.)

(a) $e^{2x} - 2e^x - 3 = 0$

(b) $3^{2x} - 4 \cdot 3^x - 12 = 0$

Example 8: Solve $\log_4(x+3) = 2$

Steps for solving logarithmic equations:

1.

2.

3.

Example 9: Solve $\log_2 x + \log_2(x + 7) = 3$

Example 10: Solve $3 \ln 2x = 12$

Example 11: Solve $\log_2(x + 2) - \log_2(x - 5) = 1$

Example 12: $\log(x + 7) - \log 3 = \log(7x + 1)$

Steps for solving equations using 1-to-1 properties:

1.

2.

3.

Example 13: $2 \log x - \log 7 = \log 112$

Example 14: $\ln(x - 3) = \ln(7x - 23) - \ln(x + 1)$

Periodic Interest Formula	Continuous Interest Formula
$A = P\left(1 + \frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$

Example 15: How long will it take \$25,000 to grow to \$500,000 at 9% interest compounded continuously?

Example 16: How long will it take \$25,000 to grow to \$500,000 at 9% interest compounded quarterly?

Example 17: What interest rate is needed for \$25,000 to double after 8 years if compounded continuously? (Round rate to nearest hundredth of a percent.)

Part II: Exponential Growth and Decay

Decay Model		Growth Model
	$A = Pe^{rt}$ $f(t) = A_0e^{kt}$ $A = A_0e^{kt}$	
	$A_0 =$ $A =$ $k =$ $t =$	

Example 18: In 2001 the world population was approximately 6.2 billion. If the annual growth rate averaged about 1.3% per year, write an exponential equation that models this situation. Use your model to estimate the population for this year.

Example 19: The **minimum wage** in 1970 was \$1.60. In 2000 it was \$5.15.

(a) Find a growth model for this situation.

Steps for finding growth/decay model:

1. Use $A = A_0 e^{kt}$ to find k .
(Estimate k to 6 decimal places.)
2. Substitute k into growth/decay model:

$$A = A_0 e^{kt}$$

(b) Estimate the minimum wage for this year.

(c) Estimate the minimum wage 10 years from now.

(d) Based on this model, when will the minimum wage be \$10.00?

Example 20: If you have an account with interest rate k , how long will it take your money to double if compounded continuously? How long will it take it to triple?

Example 21: Carbon-14 testing is used to determine the age of fossils, artifacts and paintings. Carbon-14 has a half-life of 5715 years.

(a) Find an exponential decay model for Carbon-14.

(b) A painting was discovered containing 96% of its original Carbon-14. Estimate the age of the painting.

(c) An art collector plans to purchase a painting by Leonardo DaVinci for a considerable amount of money. (DaVinci lived from 1452-1519). Could the painting in part (b) possibly be one of DaVinci's works?

Example 22: Arsenic-74 has a half-life of 17.5 days.

(a) Find an exponential decay model for this situation.

- (b) How long will it take for 5% of the original amount of arsenic-74 to remain in a blood system?
(Round to nearest day.)

- (c) What is the decay rate (per day) to the nearest tenth of a percent.

Example 23: In 2000 the population of Israel was approximately 6.04 million and by 2050 it was projected to grow to 10 million.

- (a) Find an exponential growth model for this situation.

- (b) Estimate the average annual growth rate to the nearest tenth of a percent.

Expressing an Exponential Model in Base e : $y = ab^x$ is equivalent to $y = ae^{(\ln b)x}$ or $y = ae^{x \ln b}$

Example 24: Rewrite each equation in terms of base e .

(a) $y = 68(2)^x$

(b) $y = 5000(.5)^x$

(c) $y = 2.7(0.25)^x$

(d) $y = 25(3.5)^x$

4.6 Homework Problems

Solve each equation by expressing each side as a power of the same base.

1. $3^{x+3} = 9^{x-2}$ 2. $5^{2x} = \frac{1}{\sqrt[3]{5}}$ 3. $e^x e = \frac{e^7}{e^x}$ 4. $\left(\frac{1}{3}\right)^{-3x-2} = 9^{x+1}$

Solve the exponential equations. Round answers to two decimal places.

5. $5^x = 27$ 6. $2e^{3x} = 30$ 7. $10^x + 36 = 150$ 8. $5^{x-3} = 137$
9. $e^{4x-5} - 7 = 11,243$ 10. $5^{2x+3} = 3^{x-1}$ 11. $e^{x+1} = 10^{2x}$ 12. $40e^{0.6x} - 3 = 237$

Solve by factoring. Write the answer using exact values.

13. $e^{2x} - 4e^x + 3 = 0$ 14. $e^{2x} - 3e^x + 2 = 0$ 15. $e^{2x} - 2e^x = 3$ 16. $e^{4x} + 5e^{2x} = 24$

Solve each logarithmic equation.

17. $\log_4(3x + 2) = 3$ 18. $5\ln(2x) = 20$
19. $\log_5 x + \log_5(4x - 1) = 1$ 20. $\log_3(x - 5) + \log_3(x + 3) = 2$
21. $\log(x + 4) - \log x = \log 4$ 22. $\log(5x + 1) = \log(2x + 3) + \log 2$
23. $\log(x + 4) - \log 2 = \log(5x + 1)$ 24. $\ln(x - 4) + \ln(x + 1) = \ln(x - 8)$
25. How long will it take \$2000 to grow to \$10,000 at 5% if compounded (a) continuously and (b) compounded quarterly.
26. What interest rate is needed for \$1000 to double after 10 years if compounded continuously?
27. The formula $A = 22.9e^{0.0183t}$ models the population of Texas, A , in millions, t years after 2005.
(a) What was the population in Texas in 2005? (b) Estimate the population in 2012? (c) When will the population reach 27 million?
28. You have \$2300 to invest. What interest rate is needed for the investment to grow to \$3000 in two years if the investment is compounded quarterly?
29. What interest is needed for an investment to triple in four years if it is compounded (a) semiannually and (b) continuously?
30. How much money would you need to invest now in order to have \$15,000 saved in two years if the principal is invested at an interest rate of 6.2% compounded (a) quarterly and (b) continuously?
31. A certain type of radioactive iodine has a half-life of 8 days.
(a) Find an exponential decay model, $A = A_0 e^{kt}$, for this type of iodine.
Round the k value in your formula to six decimal places.
(b) Use your model from part (a) to determine how long it will take for a sample of this type of radioactive iodine to decay to 10% of its original amount.
Round your final answer to the nearest whole day.

32. The population in Tanzania in 1987 was about 24.3 million, with an annual growth rate of 3.5%. If the population is assumed to change continuously at this rate.
- (a) estimate the population in 2008.
 - (b) in how many years after 1987 will the population be 30,000,000?
33. The half-life of aspirin in your bloodstream is 12 hours. How long will it take for the aspirin to decay to 70% of the original dosage?
34. The growth model $A = 107.4e^{0.012t}$ describes Mexico's population, A , in millions t years after 2006.
- (a) What is Mexico's growth rate?
 - (b) How long will it take Mexico to double its population?

4.6 Homework Answers: 1. 7 2. $-\frac{1}{6}$ 3. 3 4. 0 5. {2.05} 6. {.90} 7. {2.06} 8. {6.06}

9. {3.58} 10. {-2.80} 11. {.28} 12. {2.99} 13. {0, ln 3} 14. {0, ln 2} 15. {ln 3}

16. $\left\{\frac{\ln 3}{2}\right\}$ 17. $\left\{\frac{62}{3}\right\}$ 18. $\left\{\frac{e^4}{2}\right\}$ 19. $\left\{\frac{5}{4}\right\}$ 20. {6} 21. $\left\{\frac{4}{3}\right\}$ 22. {5} 23. $\left\{\frac{2}{9}\right\}$ 24. ϕ

25. (a) 32.19 years; (b) 32.39 years 26. 6.93% 27. (a) 22.9 million; (b) 26.03 million; (c) 2014

28. 13.5% 29. (a) 29.44%; (b) 27.47% 30. (a) \$13,263.31; (b) \$13,250.70 31. (a) $A = A_0 e^{-0.086643t}$

(b) 27 days 32. (a) 50.68 million; (b) 6.02 years 33. 6.2 hours 34. (a) 1.2%; (b) about 58 years