3.1 EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

What You Should Learn

- Recognize and evaluate exponential functions with base *a*.
- Graph exponential functions and use the One-to-One Property.
- Recognize, evaluate, and graph exponential functions with base *e*.
- Use exponential functions to model and solve real-life problems.

Exponential Functions

Transcendental Functions

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

$$f(x) = 4^x$$

$$f(3) = 4^3 = 64 \qquad f\left(\frac{1}{2}\right) = 4^{\frac{1}{2}}$$

Exponential Functions

$$a^{\sqrt{2}} \quad (\text{where } \sqrt{2} \approx 1.41421356)$$

as the number that has the successively closer approximations

$$a^{1.4},\ a^{1.41},\ a^{1.414},\ a^{1.4142},\ a^{1.41421},\ \ldots .$$

Example 1 – *Evaluating Exponential Functions*

Use a calculator to evaluate each function at the indicated value of *x*.

Function	*Value*
a. $f(x) = 2^x$	$x = -3.1$
b. $f(x) = 2^{-x}$	$x = \pi$
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$

Example 1 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-3.1) = 2^{-3.1}$	2 [^] [(−)] 3.1 [ENTER]	0.1166291
b. $f(\pi) = 2^{-\pi}$	2 [^] [(−)] π [ENTER]	0.1133147
c. $f\left(\frac{3}{2}\right) = 0.6^{3/2}$	.6 [^] [(] 3 [÷] 2 [)] [ENTER]	0.4647580

Graphs of Exponential Functions

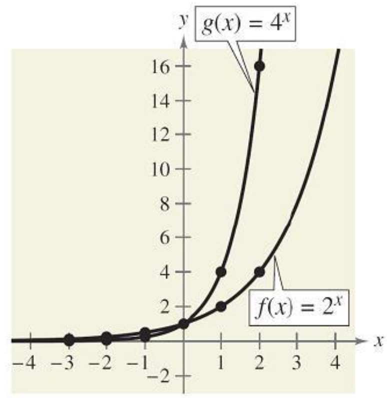
Example 2 – *Graphs of $y = a^x$*

In the same coordinate plane, sketch the graph of each function.

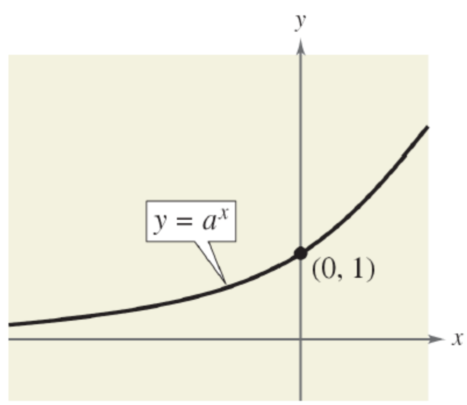
a. $f(x) = 2^x$

b. $g(x) = 4^x$

x	-3	-2	-1	0	1	2
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
4^x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

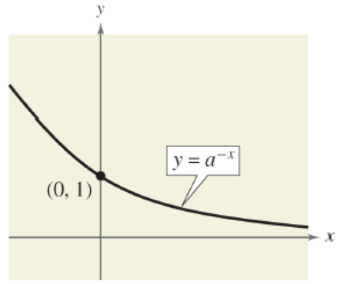


Graph of $y = a^x$, $a > 1$



- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- *y*-intercept: $(0, 1)$
- Increasing
- *x*-axis is a horizontal asymptote ($a^x \to 0$, as $x \to -\infty$).
- Continuous

Graph of $y = a^{-x}$, $a > 1$



- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: (0, 1)
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \rightarrow 0$, as $x \rightarrow \infty$).
- Continuous

For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.

One-to-One Property

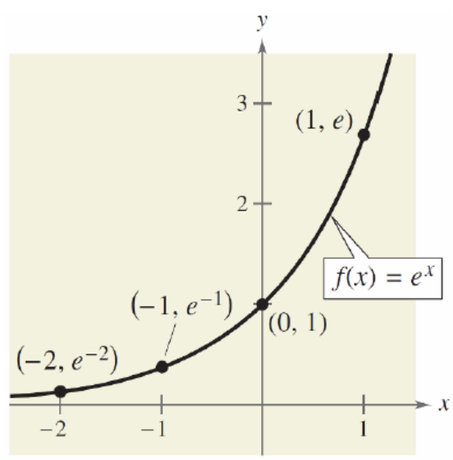
The Natural Base e

natural base

$e \approx 2.718281828 \ldots.$

natural exponential function

$f(x) = e^x$



Example 6 – *Evaluating the Natural Exponential Function*

Use a calculator to evaluate the function given by $f(x) = e^x$ at each indicated value of x.

a. $x = -2$

b. $x = -1$

c. $x = 0.25$

d. $x = -0.3$

Example 6 – *Solution*

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-2) = e^{-2}$	e^x (–) 2 ENTER	0.1353353
b. $f(-1) = e^{-1}$	e^x (–) 1 ENTER	0.3678794
c. $f(0.25) = e^{0.25}$	e^x 0.25 ENTER	1.2840254
d. $f(-0.3) = e^{-0.3}$	e^x (–) 0.3 ENTER	0.7408182

Applications

Applications: Continuously Compounded Interest

Suppose a principal P is invested at an annual interest rate r, compounded once per year. If the interest is added to the principal at the end of the year, the new balance P_1 is

$$P_1 = P + Pr$$

$$= P(1 + r)$$

Year	*Balance After Each Compounding*
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
⋮	⋮
t	$P_t = P(1 + r)^t$

Applications

Let n be the number of compoundings per year and let t be the number of years.

Then the rate per compounding is r/n and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Amount (balance) with n compoundings per year

If you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.**

Applications

In the formula for *n* compoundings per year, let *m* = *n*/*r*. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Amount with *n* compoundings per year

$$= P\left(1 + \frac{r}{mr}\right)^{mrt}$$ Substitute *mr* for *n*.

$$= P\left(1 + \frac{1}{m}\right)^{mrt}$$ Simplify.

$$= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$ Property of exponents

Applications

As m increases without bound, the table below shows that $[1 + (1/m)]^m \rightarrow e$ as $m \rightarrow \infty$.

From this, you can conclude that the formula for continuous compounding is

$A = Pe^{rt}$. Substitute e for $(1 + 1/m)^m$.

m	$\left(1 + \frac{1}{m}\right)^m$
1	2
10	2.59374246
100	2.704813829
1,000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693
↓	↓
∞	e

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Example 8 – *Compound Interest*

A total of $12,000 is invested at an annual interest rate of 9%. Find the balance after 5 years if it is compounded

a. quarterly.

b. monthly.

c. continuously.

Example 8(a) – *Solution*

For quarterly compounding, you have n = 4. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.09}{4}\right)^{4(5)}$$ Substitute for P, r, n, and t.

$$\approx \$18{,}726.11.$$ Use a calculator.

For monthly compounding, you have n = 12. So, in 5 years at 9%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.09}{12}\right)^{12(5)}$$ Substitute for P, r, n, and t.

$$\approx \$18{,}788.17.$$ Use a calculator.

For continuous compounding, the balance is

$A = Pe^{rt}$ — Formula for continuous compounding

$= 12{,}000e^{0.09(5)}$ — Substitute for P, r, and t.

$\approx \$18{,}819.75.$ — Use a calculator.