Special Factoring Formulas

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In this presentation we will be studying special factoring formulas for:

A) Perfect Square Trinomials

B) The difference of two squares

C) The sum or difference of two cubes.

A) Perfect Square Trinomials

Perhaps you remember the formula for squaring a binomial:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

The right side of each of the above equations is called a "perfect square trinomial." You can factor a perfect square trinomial by trial and error but, if you recognize the form, you can do the job easier and faster by using the factoring formula.

To factor a perfect square trinomial you look for certain clues:

1. Is the first term a perfect square of the form a^2 for some expression a?

2. Is the last term a perfect square of the form b^2 for some expression b?

3. Is the middle term plus or minus 2 times the square root of a^2 times the square root of b^2?

For example: Is the following a perfect square trinomial?

$$x^2 + 2x + 1$$

$$x^2 + 2x + 1$$

Is this term a perfect square? Yes.

Is this term a perfect square? Yes.

Is the middle term plus or minus 2 times the square root of x^2 times the square root of 1^2?

x times 1 times 2 = $2x$ yes.

Then this is a perfect square and factors into $(x + 1)^2$

See if you can recognize perfect square trinomials.

1. $x^2 + 10x + 25$ Yes, this is $(x + 5)^2$

2. $x^2 + 4x + 4$ Yes, this is $(x + 2)^2$

3. $9y^2 - 12y + 4$ Yes, this is $(3y - 2)^2$

$$\sqrt{9y^2} = 3y \ldots \sqrt{4} = 2$$

$$2(3y)(2) = 12y$$

4. $y^2 - 12y - 36$ No, -36 is not a perfect square

Is the middle term 2 times the square root of a^2 times the square root of b^2

This point concerning the middle term is important.
Consider the following example:

Factor $4x^2 - 14x + 9$.
The first term and the last term are both perfect squares but the middle term is not true to form.
2(2)(3) = 12 or –12 should be the middle term.
By trial and error $4x^2 - 14x + 9 = (4x-1)(x-9)$.

We could have a perfect square trinomial by changing the middle term:

$$4x^2 - 12x + 9 = (2x-3)(2x-3) = (2x-3)^2.$$

Try the following four problems. Watch for the pattern. Do all four before you check the answers on the next slide.

$$x^2 + 16x + 64$$

$$x^2 - 13x + 36$$

$$9x^2 - 48x + 64$$

$$4x^2 + 28x + 49$$

The solutions are:

$$x^2 + 16x + 64 = (x+8)(x+8) = (x+8)^2$$

$$x^2 - 13x + 36 = (x-9)(x-4)$$

$$9x^2 - 48x + 64 = (3x-8)(3x-8) = (3x-8)^2$$

$$4x^2 + 28x + 49 = (2x+7)(2x+7) = (2x+7)^2$$

The second problem was not a perfect square but was factored as a simple trinomial. The others were perfect square trinomials.

Try four more. Watch for the pattern for perfect square trinomials: the first term and the last term are perfect squares and the middle term is plus or minus 2ab. If your trinomial is not a perfect square, proceed by trial and error.

$$x^2 + 20x + 100$$

$$16x^2 - 40x + 25$$

$$9x^2 - 6x + 1$$

$$x^2 + 10x + 16$$

Do all four before you go to the next slide.

Here are the solutions.

$$x^2+20x+100=(x+10)(x+10)=(x+10)^2$$

$$16x^2-40x+25=(4x-5)(4x-5)=(4x-5)^2$$

$$9x^2-6x+1=(3x-1)(3x-1)=(3x-1)^2$$

$$x^2+10x+16=(x+2)(x+10)$$

The first three trinomials were perfect squares. Note that in each case, twice the product of the terms in the answer is exactly the middle term of the trinomial. The last trinomial is not a perfect square trinomial.

B) The Difference of Two Squares

In the difference of two squares

$$\underbrace{x^2 - y^2}_{\text{This side is expanded}} = \underbrace{(x-y)(x+y)}_{\text{This side is factored.}}$$

x (in the factored side) is the square root of x^2 and y (in the factored side) is the square root of y^2

Factoring the difference of two squares: some examples.

$$x^2 - 25 = (x+5)(x-5)$$

$$16 - 9b^2 = (4+3b)(4-3b)$$

$$25y^2 - 64 = (5y+8)(5y-8)$$

$$x^2 + 36 = ?$$

Note: The sum of two squares is not factorable.

Why does this binomial not factor?

$$x^2 + 36$$

Suppose it did. What would be the possible factors?

$$(x+6)(x+6) = x^2 + 12x + 36 \quad \text{No}$$

$$(x-6)(x-6) = x^2 - 12x + 36 \quad \text{No}$$

$$(x+6)(x-6) = x^2 - 36 \quad \text{No}$$

There are no other possibilities, thus it doesn't factor. In general, the sum of two squares does not factor.

Factoring the difference of two squares: Write out your answers before you go to the next page.

$$x^2 - 100 =$$

$$9y^2 - 25 =$$

$$64 - 81x^2 =$$

$$49x^2 - 1 =$$

The solutions are:

$$x^2 - 100 = (x+10)(x-10)$$

$$9y^2 - 25 = (3y+5)(3y-5)$$

$$64 - 81x^2 = (8+9x)(8-9x)$$

$$49x^2 - 1 = (7x+1)(7x-1)$$

Factoring the difference of two squares:
Write down your solutions before going to the next page.

$$x^2 - 16 =$$

$$9 - 64y^2 =$$

$$4x^2 - 49 =$$

$$9x^2 - 100 =$$

Factoring the difference of two squares: Here are the solutions.

$$x^2 - 16 = (x+4)(x-4)$$

$$9 - 64y^2 = (3+8y)(3-8y)$$

$$4x^2 - 49 = (2x+7)(2x-7)$$

$$9x^2 - 100 = (3x+10)(3x-10)$$

C) The sum or difference of two cubes

Finally, we look at two special products involving cubes.

T**he sum of two cubes**:

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

The difference of two cubes:

$$a^3 - b^3 = (a+b)\left(a^2 + ab + b^2\right)$$

In order to use these two formulas, you must be able to recognize numbers that are perfect cubes.

1000 is a perfect cube since $1000 = 10^3$

125 is a perfect cube since $125 = 5^3$

64 is a perfect cube since $64 = 4^3$

8 is a perfect cube since $8 = 2^3$

1 is a perfect cube since $1 = 1^3$

The following can be factored as the difference of two cubes: $8x^3 - 27$ letting a = 2x and b = 3

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$(2x)^3 - 3^3 = (2x-3)(4x^2 + 6x + 9)$$

Let's check with multiplication to see if the factors are correct:

$$(2x-3)(4x^2 + 6x + 9) = 8x^3 + 12x^2 + 18x - 12x^2 - 18x - 27 =$$

$$8x^3 - 27$$

For another example, the expression x^3+64 can be factored as the sum of two cubes using $a^3+b^3=(a+b)(a^2-ab+b^2)$ with $a=x$ and $b=4$.

$$x^3+64=(x+4)(x^2-4x+16)$$

To check this answer with multiplication,

$$(x+4)(x^2-4x+16)=x^3-4x^2+16x+4x^2-16x+64$$

$$=x^3+64.$$

A way to remember the formula

Clues to remember the formula:

1. Cubes always factor into a binomial times a trinomial.

2. The binomial in the factored version always contains the cube roots of the original expression with the same sign that was used in the original expression.

$$a^3 + b^3 = (a + b)(\qquad\qquad\qquad)$$

Cube root of a and b
with same sign

$$a^3 - b^3 = (a - b)(\qquad\qquad\qquad)$$

Cube root of a and b
with same sign

A way to remember the formula

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Next you use the binomial to build your trinomial:

1)Square first term

2)Find the product of both terms and change the sign i.e. a(b) =ab change the sign = -ab

3)Square last term i.e. last term is b^2

The difference of two cubes:

A binomial times a trinomial

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Cube root of a and b with same sign

1)Square first term

2)Find the product of both terms and change the sign

3)Square last term

Cube Numbers you will find in problems.

$1^3=1$

$2^3=8$

$4^3=64$

$3^3=27$

$5^3=125$

$etc^3=etc$

Difference of two cubes

A binomial times a trinomial

$$8x^3 - 125 = (2x - 5)(4x^2 + 10x + 25)$$

Cube root of 1st and 2nd term with same sign

Build trinomial with binomial.

1)Square first term

2)Find the product of both terms and change the sign

3)Square last term

Sum or Difference of two cubes

A binomial times a trinomial

$64x^3 + 1 = (4x + 1)(16x^2 - 4x + 1)$

Build trinomial

with binomial.

Now try the following four problems on your own. The answers are on the next page. Factor each of the following as the sum of two cubes or as the difference of two cubes.

$x^3 - 27$

$8x^3 + 125$

$64 - y^3$

$x^3 + 1000$

Here are the solutions to the four factoring problems with the formula substitutions for a and b at the end.

$$x^3 - 27 = (x-3)(x^2+3x+9)$$

$$a = x, b = 3$$

$$8x^3 + 125 = (2x+5)(4x^2 - 10x + 25)$$

$$a = 2x, b = 5$$

$$64 - y^3 = (4-y)(16+4y+y^2)$$

$$a = 4, b = y$$

$$x^3 + 1000 = (x+10)(x^2 - 10x + 100)$$

$$a = x, b = 10$$

Now try four more problems on your own. The answers are on the next page. Factor each of the following as the sum of two cubes or as the difference of two cubes.

$$64x^3 - 27$$

$$x^3 + 125$$

$$64x^3 + 1$$

$$27 - 125x^3$$

Here are the solutions to the four factoring problems on the previous page.

$$64x^3 - 27 = (4x-3)(16x^2+12x+9)$$

$$x^3 + 125 = (x+5)(x^2-5x+25)$$

$$64x^3 + 1 = (4x+1)(16x^2-4x+1)$$

$$27 - 125x^3 = (3-5x)(9+15x+25x^2)$$

This is the end of the unit on special factoring. At this point you should move on to the practice test for special factoring. For the exam, you will need to memorize the following formulas:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$