AP Calculus BC Formulas, Definitions, Concepts & Theorems to Know

Def. of *e*: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $e^a = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$

Absolute Value:

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition of Derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Alternative form of Def of Derivative:

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Definition of Continuity:

f is continuous at c if and only if

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$$

Differentiability:

A function f is not differentiable at $x=a$ if

1) f is not continuous at $x=a$
2) f has a cusp at $x=a$
3) f has a vertical tangent at $x=a$

Euler's Method:

Used to approximate a value of a function, given dy/dx and (x_0, y_0)

Use $y - y_0 = \frac{dy}{dx}(x-x_0)$ repeatedly.

Average Rate of Change of $f(x)$ on $[a,b]$

is the slope: $\frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}\int_a^b f'(x)\,dx$

Instantaneous Rate of Change of $f(x)$ with respect to x is $f'(x)$.

Intermediate Value Theorem (IVT):

If f is continuous on $[a,b]$ and k is any number between $f(a)$ and $f(b)$ then there is at least one number c between a and b such that $f(c)=k$

Mean Value Theorem (MVT):

If f is continuous on $[a,b]$ and differentiable on (a,b) then there exists a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. (Think: The slope at $x=c$ is the same as the slope from a to b.)

Trig Identities to Know:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$

$\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos 2x$

Definition of a Definite Integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(a+\frac{(b-a)i}{n}\right)\cdot\left(\frac{b-a}{n}\right)\right]$$

Also know Riemann Sums –

Left, Right, Midpoint, Trapezoidal

Average Value of a function $f(x)$ on $[a,b]$:

$$f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Curve Length of $f(x)$ on $[a,b]$:

$$L = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

Logistic Differential Equation:

$$\frac{dP}{dt} = kP(L-P)\ ;\ P(t) = \frac{L}{1+ce^{-Lkt}}\ ,\ c = \frac{L-P(0)}{P(0)}$$

Derivatives

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\cdot g'(x)$$

(chain rule)

$$\frac{d}{dx}(uv)=u'v+uv'$$

(product rule)

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$$

(quotient rule)

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

(power rule)

$$\left(f^{-1}\right)'(a)=\frac{1}{f'\left(f^{-1}(a)\right)}$$

(derivative of an inverse)

$$\frac{d}{dx}(\sin u)=\cos u\cdot u'$$

$$\frac{d}{dx}(\cos u)=-\sin u\cdot u'$$

$$\frac{d}{dx}(\tan u)=\sec^2 u\cdot u'$$

$$\frac{d}{dx}(\sec u)=\sec u\tan u\cdot u'$$

$$\frac{d}{dx}(\cot u)=-\csc^2 u\cdot u'$$

$$\frac{d}{dx}(\csc u)=-\csc u\cot u\cdot u'$$

$$\frac{d}{dx}(\ln u)=\frac{u'}{u}$$

$$\frac{d}{dx}(\log_b u)=\frac{u'}{u}\left(\frac{1}{\ln b}\right)$$

$$\frac{d}{dx}\left(e^u\right)=u'e^u$$

$$\frac{d}{dx}\left(a^u\right)=u'a^u\ln a$$

$$\frac{d}{dx}(\arcsin u)=\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\arctan u)=\frac{u'}{1+u^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} u)=\frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}(\arccos u)=-\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\operatorname{arccot} u)=-\frac{u'}{1+u^2}$$

$$\frac{d}{dx}(\operatorname{arccsc} u)=-\frac{u'}{|u|\sqrt{u^2-1}}$$

Integrals

$\int u^n du = \frac{u^{n+1}}{n+1} + c$

(power rule for integrals)

$\int \frac{1}{u} du = \ln|u| + C$

$\int e^u du = e^u + C$

$\int a^u du = \frac{a^u}{\ln a} + C$

$\int \ln u du = u \ln u - u + C$

$\int \cos u du = \sin u + C$

$\int \sin u du = -\cos u + C$

$\int \sec^2 u du = \tan u + C$

$\int \sec u \tan u du = \sec u + C$

$\int \csc^2 u du = -\cot u + C$

$\int \csc u \cot u du = -\csc u + C$

$\int \tan u du = -\ln|\cos u| + C$

$\int \sec u du = \ln|\sec u + \tan u| + C$

$\int \cot u du = \ln|\sin u| + C$

$\int \csc u du = -\ln|\csc u + \cot u| + C$

$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$

Techniques of Integration:

u-substitution; Partial Fractions; Completing the Square; Integration By-Parts: $\int u\,dv = uv - \int v\,du$

Max/Min, Concavity, Inflection Point

Critical Number at $x = c$ if:

$f'(c) = 0$ or $f'(c)$ is undefined

First Derivative Test:

Let c be a critical number.

If $f'(x)$ changes from $+$ to $-$ at $x = c$

then f has a relative max of $f(c)$.

If $f'(x)$ changes from $-$ to $+$ at $x = c$

then f has a relative min of $f(c)$

Second Derivative Test:

If $f'(c) = 0$ and $f''(c) > 0$

then f has a relative min of $f(c)$.

If $f'(c) = 0$ and $f''(c) < 0$

then f has a relative max of $f(c)$.

Absolute Maxima:

The absolute max on a closed interval $[a, b]$ is

$f(a)$, $f(b)$, or a relative maximum.

The absolute min on a closed interval $[a, b]$ is

$f(a)$, $f(b)$, or a relative minimum.

Test for Concavity:

If $f''(x) > 0$ for all x in I , then the graph of f is concave up on I .

If $f''(x) < 0$ for all x in I , then the graph of f is concave down on I .

Inflection Point:

A function has an inflection point at $(c, f(c))$ if

f'' changes sign at $x = c$

Fundamental Theorems

First Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

(The **accumulated change** in f from a to b)

Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$ or

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x) - f(h(x))h'(x)$$

Area & Volume (Functions in the form $y = f(x)$ or $x = f(y)$)

Area Between Curves

$$A = \int_a^b topcurve - bottomcurve\ dx$$

$$A = \int_c^d rightcurve - leftcurve\ dy$$

Volume – General Volume Formula

$$Volume = \int_a^b A(x)\ dx, \text{ where } A(x) = area$$

$$Volume = \int_c^d A(y)\ dy, \text{ where } A(y) = area$$

Volume – Disc/Washer Method

$$V = \pi \int_a^b \left(R(x)\right)^2 - \left(r(x)\right)^2 dx$$

$$V = \pi \int_c^d \left(R(y)\right)^2 - \left(r(y)\right)^2 dy$$

Volume – Cylindrical Shell Method

$$V = 2\pi \int_a^b (radius)(height)\,dx$$

$$V = 2\pi \int_c^d (radius)(height)\,dy$$

Note that the Shell Method is NOT tested on the AP Exam.

Volume by Cross-Sections

⊥ x-axis: $V = \int_a^b A(x)\,dx$

⊥ y-axis: $V = \int_a^b A(y)\,dy$

$A_{semicircle} = \frac{\pi}{8}s^2$; $A_{RtIsosc\Delta} = \frac{1}{2}s^2$ (leg as s)

$A_{Equil\Delta} = \frac{\sqrt{3}}{4}s^2$; $A_{RtIsosc\Delta} = \frac{1}{4}s^2$ (hypot as s)

Horizontal/Vertical Motion

Position Function: $s(t)$

Velocity Function: $v(t) = s'(t)$

Acceleration Function: $a(t) = v'(t) = s''(t)$

Displacement (change in position) over $[a, b]$

$$= \int_a^b v(t)\,dt = s(b) - s(a)$$

Total Distance Traveled over $[a, b]$ $= \int_a^b |v(t)|\,dt$

Speed = $|v(t)|$

Speed Increases if $v(t)$ and $a(t)$ have same sign

Speed Decreases if $v(t)$ and $a(t)$ different signs

Motion Along a Curve (Parametrics & Vectors)

Position Vector $=\langle x(t), y(t)\rangle$

Velocity Vector $=\langle x'(t), y'(t)\rangle$

Acceleration Vector $=\langle x''(t), y''(t)\rangle$

Slope$=\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\qquad \dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]=\dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

|**Displacement**| $=\sqrt{\left(x(b)-x(a)\right)^2+\left(y(b)-y(a)\right)^2}$

Speed (or **Magnitude/Length of Velocity Vector**)

$$=\left|\vec{v}(t)\right|=\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}$$

Speed Increases if $\dfrac{d}{dt}(speed)>0$

Distance Traveled (or **Length of Curve**)

$$=\int_a^b\left|\vec{v}(t)\right|dt=\int_a^b\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$$

Polar Curves

$x = r\cos\theta$, $y = r\sin\theta$

Slope of polar curve: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

Area inside a polar curve: $A = \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

Series

Convergence/Divergence of Series

10 Tests: nth Term Test, Telescoping Series Test, Geometric Series Test, p-Series Test, Integral Test, Direct Comparison Test, Limit Comparison Test, Alternating Series Test, Ratio Test, and Root Test.

Alternating Series Error Bound

If a series is alternating in sign and decreasing in magnitude, and to zero, then

$$|\text{error}| \le |\text{first disregarded term}|$$

Taylor Series

$$P(x) = f(c) + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \ldots$$

is called the nth degree Taylor Series for $f(x)$, centered at $x = c$.

Lagrange Error Bound (aka **Taylor's Theorem**)

$$|f(x) - P_n(x)| = |R_n(x)| \le \left| \frac{\max f^{(n+1)}(z) \cdot (x-c)^{n+1}}{(n+1)!} \right|$$

Power Series of e^x, $\sin x$, $\cos x$, centered at $x = 0$

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

e^x, $\sin x$, and $\cos x$ converge for all real x-values

Hyperbolic Trig Functions are not part of the curriculum for AP Calculus BC. They are used in Differential Equations and other math courses, as well as in some of the sciences. Here is a crash-course on hyperbolic functions. You'll notice many similarities to basic trig functions.

Definitions of **Hyperbolic Trig Functions**:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Identities involving hyperbolic trig functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^2 x = -\frac{1}{2} + \frac{1}{2}\cosh(2x)$$

$$\cosh^2 x = \frac{1}{2} + \frac{1}{2}\cosh(2x)$$

Derivatives of hyperbolic trig functions:

$$\frac{d}{dx}\sinh u = \cosh u \cdot u'$$

$$\frac{d}{dx}\cosh u = \sinh u \cdot u'$$

$$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \cdot u'$$

$$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot u'$$

$$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \cdot u'$$

$$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \cdot u'$$

Integrals involving hyperbolic trig functions:

$$\int \cosh u\,du = \sinh u + C$$

$$\int \sinh u\,du = \cosh u + C$$

$$\int \operatorname{sech}^2 u\,du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\,du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$$

Inverses of hyperbolic trig functions:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad D(-\infty,\infty)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad D(1,\infty)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad D(-1,1)$$

$$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1} \quad D:(-\infty,1)\cup(1,\infty)$$

$$\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x} \quad D:(0,1]$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) \quad D:(-\infty,0)\cup(0,\infty)$$

Derivatives of inverse hyperbolic trig functions:

$$\frac{d}{dx}\sinh^{-1} u = \frac{u'}{\sqrt{u^2+1}}$$

$$\frac{d}{dx}\cosh^{-1} u = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}\tanh^{-1} u = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-u'}{|u|\sqrt{1-u^2}}$$

$$\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\frac{d}{dx}\coth^{-1} u = \frac{u'}{1-u^2}$$