CHAPTER 37
Infinite Series

37.1 Prove that, if $\Sigma\, a_n$ converges, then $\lim_{n\to+\infty} a_n = 0$.

▌ Let $S = \sum_{n=0}^{\infty} a_n$. Then $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \to S - S = 0$.

37.2 Show that the harmonic series $\Sigma\, 1/n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges.

▌ $1 > \frac{1}{2}$, $\frac{1}{2} \ge \frac{1}{2}$, $\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$, $\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$, etc. Therefore, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \to +\infty$. [Alternatively, by the integral test, $\int_1^{\infty} \frac{1}{x}\, dx = \lim_{u\to+\infty} \int_1^{u} \frac{1}{x}\, dx = \lim_{u\to+\infty} (\ln x\,]_1^u) = \lim_{u\to+\infty} \ln u = +\infty$.]

37.3 Does $\lim_{n\to+\infty} a_n = 0$ imply that $\Sigma\, a_n$ converges?

▌ No. The harmonic series $\Sigma\, 1/n$ (Problem 37.2) is a counterexample.

37.4 Let $S_n = a + ar + \cdots + ar^{n-1}$, with $r \ne 1$. Show that $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

▌ $rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$. $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Hence, $(r-1)S_n = ar^n - a = a(r^n - 1)$. Thus, $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

37.5 Let $a \ne 0$. Show that the infinite geometric series $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1 - r}$ if $|r| < 1$ and diverges if $|r| \ge 1$.

▌ By Problem 37.4, $S_n = \dfrac{a(r^n - 1)}{r - 1}$. If $|r| < 1$, $S_n \to \dfrac{a(-1)}{r - 1} = \dfrac{a}{1 - r}$, since $r^n \to 0$; if $|r| > 1$, $|S_n| \to +\infty$, since $|r|^n \to +\infty$. If $r = 1$, the series is $a + a + a + \cdots$, which diverges since $a \ne 0$. If $r = -1$, the series is $a - a + a - a + \cdots$, which oscillates between a and 0.

37.6 Evaluate $\sum_{n=0}^{\infty} \dfrac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

▌ By Problem 37.5, with $r = \dfrac{1}{2}$, $\sum_{n=0}^{\infty} \dfrac{1}{2^n} = \dfrac{1}{1 - \frac{1}{2}} = 2$.

37.7 Evaluate $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n} = 1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$.

▌ By Problem 37.5, with $r = -\dfrac{1}{3}$, $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n} = \dfrac{1}{1 - (-\frac{1}{3})} = \dfrac{3}{4}$.

37.8 Show that the infinite decimal $0.9999\cdots$ is equal to 1.

▌ $0.999\cdots = \dfrac{9}{10} + \dfrac{9}{10^2} + \dfrac{9}{10^3} + \cdots = \dfrac{\frac{9}{10}}{1 - \frac{1}{10}} = 1$, by Problem 37.5, with $r = \frac{1}{10}$.

37.9 Evaluate the infinite repeating decimal $d = 0.215626262\cdots$.

▌ $d = 0.215 + \dfrac{62}{10^5} + \dfrac{62}{10^7} + \dfrac{62}{10^9} + \cdots$. By Problem 37.5, with $r = \dfrac{1}{10^2}$, $\dfrac{62}{10^5} + \dfrac{62}{10^7} + \dfrac{62}{10^9} + \cdots = \dfrac{62/10^5}{1 - 1/10^2} = \dfrac{62}{99{,}000}$. Hence, $d = \dfrac{215}{1000} + \dfrac{62}{99{,}000} = \dfrac{21{,}347}{99{,}000}$.

37.10 Investigate the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots$.

▮ $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$. Hence, the partial sum

$$S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}\to 1.$$

The series converges to 1. (The method used here is called "telescoping.")

37.11 Study the series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}+\cdots$.

▮ $\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$. So $S_n=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)\to\frac{1}{2}$. Thus, the series converges to $\frac{1}{2}$.

37.12 Find the sum of the series $4-1+\frac{1}{4}-\frac{1}{16}+\cdots$.

▮ This is a geometric series with ratio $r=-\frac{1}{4}$ and first term $a=4$. Hence, it converges to $\frac{4}{1-(-\frac{1}{4})}=\frac{16}{5}$.

37.13 Test the convergence of $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\cdots$.

▮ This is a geometric series with ratio $r=\frac{3}{2}>1$. Hence, it is divergent.

37.14 Test the convergence of $3+\frac{5}{2}+\frac{7}{3}+\frac{9}{4}+\cdots$.

▮ The series has the general term $a_n=\frac{2n+3}{n+1}$ (starting with $n=0$), but $\lim a_n=\lim\frac{2+3/n}{1+1/n}=2\neq 0$. Hence, by Problem 37.1, the series diverges.

37.15 Investigate the series $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\cdots$.

▮ Rewrite the series as $\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots\right)+\frac{1}{2^2}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\right)=\frac{1}{2}+\frac{1}{4}(1)=\frac{3}{4}$, by Problems 37.11 and 37.10.

37.16 Test the convergence of $3+\sqrt{3}+\sqrt[3]{3}+\sqrt[4]{3}+\cdots$.

▮ $a_n=\sqrt[n]{3}=3^{1/n}=e^{(\ln 3)/n}\to e^0=1\neq 0$. Hence, by Problem 37.1, the series diverges.

37.17 Study the series $\sum_{n=1}^{\infty}\frac{1}{n(n+4)}=\frac{1}{1\cdot 5}+\frac{1}{2\cdot 6}+\frac{1}{3\cdot 7}+\cdots$.

▮ $\frac{1}{n(n+4)}=\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+4}\right)$. So the partial sum $S_n=\frac{1}{4}\left(\frac{1}{1}-\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{2}-\frac{1}{6}\right)+\cdots+\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+4}\right)=\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{4}\left(-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right)\to\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{25}{48}$.

37.18 Study the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots$.

▮ $\frac{1}{n(n+1)(n+2)}=\frac{1}{n(n+2)}-\frac{1}{(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+2}\right)-\frac{1}{n+1}$. Thus, $\frac{1}{1\cdot 2\cdot 3}=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{3}\right)-\frac{1}{2}$, $\frac{1}{2\cdot 3\cdot 4}=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{3}$, $\frac{1}{3\cdot 4\cdot 5}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}\right)-\frac{1}{4},\ldots,$ $\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+2}\right)-\frac{1}{n+1}$. The partial sum $\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{n+1}\left(-\frac{1}{2}\right)+\frac{1}{n+2}\left(\frac{1}{2}\right)\to\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$.

37.19 Study the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$.

▌ $\frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$. Hence, the partial sum $\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left[\frac{1}{n!} - \frac{1}{(n+1)!}\right] = 1 - \frac{1}{(n+1)!} \to 1$.

37.20 Study the series $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots$.

▌ $\frac{1}{(4n-3)(4n+1)} = \frac{1}{4}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$. So

$$S_n = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) + \frac{1}{4}\left(\frac{1}{5} - \frac{1}{9}\right) + \cdots + \frac{1}{4}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right) = \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) \to \frac{1}{4}.$$

37.21 Evaluate $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)}$.

▌ $\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}$. So the partial sum $\left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{4} + \frac{1}{5}\right) + \cdots$ is either $1 - \frac{1}{n+1}$ or $1 - \frac{1}{n+1} + \left(\frac{1}{n+1} + \frac{1}{n+2}\right)$. In either case, the partial sum approaches 1.

37.22 Evaluate $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

▌ $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$. The partial sum $S_n = \left(\frac{1}{1} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] = 1 - \frac{1}{(n+1)^2} \to 1$.

37.23 Evaluate $\sum_{n=1}^{\infty} \frac{e^n}{n^3} = e + \frac{e^2}{8} + \frac{e^3}{27} + \frac{e^4}{64} + \cdots$.

▌ $\frac{e^n}{n^3} \to +\infty$; so, by Problem 37.1, the series diverges.

37.24 Evaluate $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}}$.

▌ $\frac{1}{\sqrt{n} + \sqrt{n-1}} = \frac{1}{\sqrt{n} + \sqrt{n-1}} \cdot \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} - \sqrt{n-1}} = \sqrt{n} - \sqrt{n-1}$. The partial sum $(\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \cdots + (\sqrt{n} - \sqrt{n-1}) = \sqrt{n} \to +\infty$. The series diverges.

37.25 Find the sum $S = \sum_{n=0}^{\infty} \frac{1}{3^n}$, and show that it is correct by exhibiting a formula which, for each $\varepsilon > 0$, specifies an integer m for which $|S_n - S| < \varepsilon$ holds for all $n > m$ (where S_n is the nth partial sum).

▌ $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with ratio $r = \frac{1}{3}$ and first term $a = 1$. So the sum $S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$. In fact, assume $\varepsilon > 0$. Then, by Problem 37.4, $S_n = 1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}} = \frac{(\frac{1}{3})^n - 1}{\frac{1}{3} - 1} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$. Now, $|S_n - S| = \left|\frac{3}{2} - \frac{3}{2}\left(1 - \frac{1}{3^n}\right)\right| = \frac{1}{2\cdot 3^{n-1}}$. We want $\frac{1}{2\cdot 3^{n-1}} < \varepsilon$, $\frac{1}{2\varepsilon} < 3^{n-1}$, $-\ln 2\varepsilon < (n-1)\ln 3$, $-\frac{\ln 2\varepsilon}{\ln 3} < n - 1$. Choose m to be the least positive integer that exceeds $-\frac{\ln 2\varepsilon}{\ln 3}$.

37.26 Determine the value of the infinite decimal $0.666 + \cdots$.

▌ $0.666\cdots = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \cdots$ is a geometric series with ratio $r = \frac{1}{10}$ and first term $a = \frac{6}{10}$. Hence, the sum is $\frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{2}{3}$.

37.27 Evaluate $\sum_{n=0}^{\infty} \frac{1}{n+100} = \frac{1}{100} + \frac{1}{101} + \cdots$.

▌ $\sum_{n=0}^{\infty} \frac{1}{n+100}$ is the harmonic series minus the first 99 terms. However, convergence or divergence is not affected by deletion or addition of any finite number of terms. Since the harmonic series is divergent (by Problem 37.2), so is the given series.

37.28 Evaluate $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \cdots$.

▌ $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$. Since the harmonic series is divergent, so is the given series.

37.29 Evaluate $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$.

▌ $\ln[n/(n+1)] = \ln n - \ln(n+1)$, and $S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + [\ln n - \ln(n+1)] = -\ln(n+1) \to -\infty$. Thus, the given series diverges.

37.30 Evaluate $\sum_{n=0}^{\infty} e^{-n} = 1 + e^{-1} + e^{-2} + \cdots$.

▌ This is a geometric series with ratio $r = 1/e < 1$ and first term $a = 1$. Hence, it converges to $\frac{1}{1 - 1/e} = \frac{e}{e-1}$.

37.31 Evaluate $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{5^n}\right)$.

▌ This series converges because it is the sum of two convergent series, $\sum_{n=0}^{\infty} \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{5^n}$ (both are geometric series with ratio $r < 1$). $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$, and $\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$. Hence, the sum of the given series is $2 + \frac{5}{4} = \frac{13}{4}$.

37.32 (Zeno's paradox) Achilles (A) and a tortoise (T) have a race. T gets a 1000-ft head start, but A runs at 10 ft/s while the tortoise only does 0.01 ft/s. When A reaches T's starting point, T has moved a short distance ahead. When A reaches that point, T again has moved a short distance ahead, etc. Zeno claimed that A would never catch T. Show that this is not so.

▌ When A reaches T's starting point, 100 s have passed and T has moved $0.01 \times 100 = 1$ ft. A covers that additional 1 ft in 0.1 s, but T has moved $0.01 \times 0.1 = 0.001$ ft further. A needs 0.0001 s to cover that distance, but T meanwhile has moved $0.01 \times 0.0001 = 0.000001$ ft; and so on. The limit of the distance between A and T approaches 0. The time involved is $100 + 0.1 + 0.0001 + 0.0000001 + \cdots$, which is a geometric series with first term $a = 100$ and ratio $r = \frac{1}{1000}$. Its sum is $100/(1 - \frac{1}{1000})$. Thus, Achilles catches up with (and then passes) the tortoise in a little over 100 s, just as we knew he would. The seeming paradox arises from the artificial division of the event into infinitely many shorter and shorter steps.

37.33 A rubber ball falls from an initial height of 10 m; whenever it hits the ground, it bounces up two-thirds of the previous height. What is the total distance covered by the ball before it comes to rest?

▌ The distance is $10 + 2[10(\frac{2}{3}) + 10(\frac{2}{3})^2 + 10(\frac{2}{3})^3 + \cdots]$. In brackets is a geometric series with ratio $\frac{2}{3}$ and first term $\frac{20}{3}$; its sum is $\frac{20}{3}/(1 - \frac{2}{3}) = 20$, for a distance of $10 + 2(20) = 50$ m.

37.34 Investigate $\sum_{n=1}^{\infty} \frac{3}{5^n + 2}$.

▌ $3/(5^n + 2) < 5/5^n = 1/5^{n-1}$. So this series of positive terms is term by term less than the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{5^{n-1}}$. Hence, by the comparison test, the given series is convergent. However, we cannot directly compute the sum of the series. We can only say that the sum is less than $\sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$.

37.35 Determine whether $\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$ is convergent.

▮ $1/(2^n - 3) < 1/2^{n-1}$ for all $n \ge 3$ (since $2^{n-1} < 2^n - 3 \Leftrightarrow 3 < 2^n - 2^{n-1} = 2^{n-1}$). Hence, beginning with the third term, the given series is term by term less than the convergent series $\sum_{n=3}^{\infty} \frac{1}{2^{n-1}}$, and, therefore, converges.

37.36 If $0 < p \le 1$, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is divergent.

▮ $1/n^p \ge 1/n$ since $n^p \le n$. Therefore, by the comparison test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

37.37 Determine whether $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent.

▮ For $n \ge 1$, $1/n! = 1/(1 \cdot 2 \cdot \cdots \cdot n) \le 1/(1 \cdot 2 \cdot 2 \cdot \cdots \cdot 2) = 1/2^{n-1}$. Hence, $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent, by comparison with the convergent series $\sum_{n=0}^{\infty} \frac{1}{2^{n-1}}$. The sum ($= e$) of the given series is $< 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + 2 = 3$.

37.38 Determine whether $\sum_{n=1}^{\infty} \frac{1}{(n^3 + 10)^{1/4}}$ is convergent.

▮ $1/(n^3 + 10) \ge 1/2n^3$ for $n \ge 3$ (since $n^3 > 10$ for $n \ge 3$). Therefore, $1/(n^3 + 10)^{1/4} \ge 1/2^{1/4} n^{3/4}$. Thus, the given series is divergent by comparison with $\frac{1}{2^{1/4}} \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ (see Problem 37.36).

37.39 State the integral test.

▮ Let $\sum a_n$ be a series of positive terms such that there is a continuous, decreasing function $f(x)$ for which $f(n) = a_n$ for all positive integers $n \ge n_0$. Then $\Sigma\, a_n$ converges if and only if the improper integral $\int_{n_0}^{\infty} f(x)\, dx$ converges.

37.40 For $p > 1$, show that the so-called p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. (Compare with Problem 37.36.)

▮ Use the integral test (Problem 37.39), with $f(x) = 1/x^p$. $\int_1^{\infty} \frac{dx}{x^p} = \lim_{u \to +\infty} \int_1^u \frac{dx}{x^p} = \lim_{u \to +\infty} \frac{1}{1-p} \frac{1}{x^{p-1}} \Big]_1^u = \lim_{u \to +\infty} \frac{1}{1-p} \left(\frac{1}{u^{p-1}} - 1 \right) = \frac{1}{p-1}$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

37.41 Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

▮ Use the integral test with $f(x) = \frac{1}{x \ln x}$.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{u \to +\infty} \int_2^u \frac{1}{x \ln x}\, dx = \lim_{u \to \infty} \ln(\ln x) \Big]_2^u = \lim_{u \to +\infty} [\ln(\ln u) - \ln(\ln 2)] = +\infty.$$

Hence, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

37.42 State the limit comparison test.

▮ Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be series of positive terms.
Case I. If $\lim_{n \to +\infty} a_n/b_n = L > 0$, then $\Sigma\, a_n$ converges if and only if $\Sigma\, b_n$ converges.
Case II. If $\lim_{n \to +\infty} a_n/b_n = 0$ and $\Sigma\, b_n$ converges, then $\Sigma\, a_n$ converges.
Case III. If $\lim_{n \to +\infty} a_n/b_n = +\infty$ and Σb_n diverges, then $\Sigma\, a_n$ diverges.

37.43 If $\Sigma\, a_n$ and $\Sigma\, b_n$ are series of positive terms and $\lim_{n\to+\infty} a_n/b_n = 0$, show by example that $\Sigma\, a_n$ may converge and $\Sigma\, b_n$ not converge.

Let $a_n = 1/n^2$ and $b_n = 1/n$. $\lim_{n\to+\infty} a_n/b_n = \lim_{n\to+\infty} 1/n = 0$. But, $\Sigma\, 1/n^2$ converges and $\Sigma\, 1/n$ diverges.

37.44 Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3}}$ converges.

Use the limit comparison test with the convergent p-series $\sum \frac{1}{n^{3/2}}$. Then $\lim_{n\to+\infty} \frac{1/\sqrt{n^3+3}}{1/n^{3/2}} = \lim_{n\to+\infty} \sqrt{\frac{n^3}{n^3+3}} = \lim_{n\to+\infty} \sqrt{\frac{1}{1+3/n^3}} = 1$. Therefore, the given series is convergent.

37.45 Determine whether $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ is convergent.

Use the comparison test: $\frac{1}{n3^n} \le \frac{1}{3^n}$. The geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent. Hence, the given series is convergent.

37.46 Determine whether $\sum_{n=1}^{\infty} \frac{n^3}{2n^4+1}$ is convergent.

Intuitively, we ignore the 1 in the denominator, so we use the limit comparison test with the divergent series $\sum 1/n$. $\lim_{n\to+\infty} \frac{n^3/(2n^4+1)}{1/n} = \lim_{n\to+\infty} \frac{n^4}{2n^4+1} = \lim_{n\to+\infty} \frac{1}{2+1/n^4} = \frac{1}{2}$. Hence, the given series is divergent.

37.47 Determine whether $\sum_{n=2}^{\infty} \frac{\ln n}{n^2+1}$ is convergent.

Use the limit comparison test with the convergent p-series $\sum 1/n^{3/2}$. $\lim_{n\to+\infty} \frac{(\ln n)/(n^2+1)}{1/n^{3/2}} = \lim_{n\to+\infty} \frac{n^{3/2}\ln n}{n^2+1} = \lim_{n\to+\infty} \frac{1}{1+1/n^2}\,\frac{\ln n}{n^{1/2}} = 0$ $\left(\text{since, by L'Hôpital's rule, } \lim_{n\to+\infty} \frac{\ln n}{n^{1/2}} = \lim_{n\to+\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = \lim_{n\to+\infty} \frac{2}{\sqrt{n}} = 0\right)$. Therefore, the given series is convergent.

37.48 Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is convergent.

$\frac{1}{n^n} = \frac{1}{e^{n\ln n}} < \frac{1}{e^n}$, for $n \ge 3$. Hence, the series converges, by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{e^n}$.

37.49 Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges.

Use the integral test with $f(x) = \frac{\ln x}{x}$. $\left[\text{Note that } f(x) \text{ is decreasing, since } f'(x) = \frac{1-\ln x}{x^2} < 0 \text{ for } x > e.\right]$ $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{u\to+\infty} \int_1^u \frac{\ln x}{x}\,dx = \lim_{u\to+\infty} \{\frac{1}{2}(\ln x)^2\,]_1^u\} = \lim_{u\to+\infty} \frac{1}{2}(\ln u)^2 = +\infty$. Therefore, the given series diverges.

37.50 Determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

Use the integral test with $f(x) = 1/x(\ln x)^2$. $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{u\to+\infty} \int_2^u \frac{dx}{x(\ln x)^2} = \lim_{u\to+\infty} \left(-\frac{1}{\ln x}\Big]_2^u\right) = \lim_{u\to+\infty} \left[-\left(\frac{1}{\ln u} - \frac{1}{\ln 2}\right)\right] = \frac{1}{\ln 2}$. Hence, the series converges.

37.51 Give an example of a series that is conditionally convergent (that is, convergent but not absolutely convergent).

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent by the alternating series test (the terms are alternately positive and negative, and their magnitudes decrease to zero). But $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

37.52 State the ratio test for a series Σa_n.

▌ Assume $a_n \neq 0$ for $n \geq n_0$.
Case I. If $\lim_{n\to+\infty} |a_{n+1}/a_n| < 1$, the series is absolutely convergent.
Case II. If $\lim_{n\to+\infty} |a_{n+1}/a_n| > 1$, the series is divergent.
Case III. If $\lim_{n\to+\infty} |a_{n+1}/a_n| = 1$, nothing can be said about convergence or divergence.

37.53 Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ is convergent.

▌ Apply the ratio test. $\lim_{n\to+\infty} |a_{n+1}/a_n| = \lim_{n\to+\infty} \frac{(n+1)^2/e^{n+1}}{n^2/e^n} = \lim_{n\to+\infty} \frac{(n+1)^2}{n^2}\frac{1}{e} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$. Hence, the series converges. (The integral test is also applicable.)

37.54 Determine whether $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ is convergent.

▌ Use the comparison test with the convergent p-series $\frac{\pi}{2}\sum \frac{1}{n^2}$. Clearly, $\frac{\tan^{-1} n}{n^2+1} < \frac{\pi/2}{n^2}$. Hence, the given series converges. (The integral test is also applicable.)

37.55 Determine whether $1 - \frac{2}{3} + \frac{3}{3^2} - \frac{4}{3^3} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{3^n}$ is convergent.

▌ By the ratio test, the series is absolutely convergent; so it is certainly convergent.

37.56 Determine whether $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2n+1}$ converges.

▌ Although the terms are alternatively positive and negative, the alternating series test does not apply, since $\lim_{n\to+\infty} \frac{n+1}{2n+1} = \lim_{n\to+\infty} \frac{1+1/n}{2+1/n} = \frac{1}{2} \neq 0$. Since the nth term does not approach 0, the series is not convergent.

37.57 Find the error if the sum of the first three terms is used as an approximation to the sum of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$.

▌ The error is less than the magnitude of the first term omitted. Thus, the approximation is $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, and the error is less than $\frac{1}{4}$. Hence, the actual value V satisfies $\frac{7}{12} < V < \frac{13}{12}$. [N.B. It can be shown that $V = \ln 2 \approx 0.693$.]

37.58 Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \cdots$ correct to three decimal places.

▌ We want an error < 0.0005. Hence, we must find the least n so that $1/n! < 0.0005 = \frac{1}{2000}$; that is, $n! > 2000$. Since $6! = 720$ and $7! = 5040$, the desired value of n is 7. So we must use $-1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = -\frac{455}{720} \approx -0.628$. [The actual value is $e^{-1} - 1$.]

37.59 Approximate the sum $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$ to two decimal places using the method based on the alternating series test. Check your answer by finding the actual sum.

▌ We must find the least n such that $1/4^n < 0.005 = \frac{1}{200}$, that is, $200 < 4^n$, or $n \geq 4$. So, if we use $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} = \frac{51}{64}$, the error will be less than 0.0005. Since $\frac{51}{64} = 0.796\cdots$, our approximation is 0.80. To check, note that the given series is a geometric series with ratio $-\frac{1}{4}$, and, therefore, the sum of the series is $\frac{1}{1-(-\frac{1}{4})} = \frac{4}{5} = 0.8$. Thus, our approximation is actually the exact value of the sum.

37.60 Estimate the error when $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ is approximated by its first 10 terms.

▌ The error is less than the magnitude of the first term omitted, which is $1/11^2 \approx 0.0083$.

37.61 How many terms must be used to approximate the sum of the series in Problem 37.60 correctly to one decimal place?

▌ We must have $1/n^2 < 0.05 = \frac{1}{20}$, or $20 < n^2$, $n \geq 5$. Thus, we must use the first four terms $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} = \frac{115}{144} \approx 0.79$. Hence, correct to one decimal place, the sum is 0.8.

37.62 Estimate the error when the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is approximated by its first 50 terms.

▌ $a_n = (-1)^{n+1}/(2n-1)$. The error will be less than the magnitude of the first term omitted: $1/[2(51)-1] = \frac{1}{101}$. Notice how poor the guaranteed approximation is for such a large number of terms.

37.63 Estimate the number of terms of the series $1 - \dfrac{1}{2^4} + \dfrac{1}{3^4} - \dfrac{1}{4^4} + \cdots$ required for an approximation of the sum which will be correct to four decimal places.

▌ We must have $1/n^4 < 0.00005 = \frac{1}{20{,}000}$, $n^4 > 20{,}000$, $n \geq 12$. Hence, we should use the first 11 terms.

37.64 Show that, if $\Sigma\, a_n$ converges by the integral test for a function $f(x)$, the error R_n, if we use the first n terms, satisfies $\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_n^{\infty} f(x)\,dx$.

▌ $R_n = a_{n+1} + a_{n+2} + \cdots$. If we approximate by the lower rectangles in Fig. 37-1, then $R_n < \int_n^{n+1} f(x)\,dx + \int_{n+1}^{n+2} f(x)\,dx + \cdots = \int_n^{\infty} f(x)\,dx$. If we use the upper rectangles, $R_n > \int_{n+1}^{n+2} f(x)\,dx + \int_{n+2}^{n+3} f(x)\,dx + \cdots = \int_{n+1}^{\infty} f(x)\,dx$.

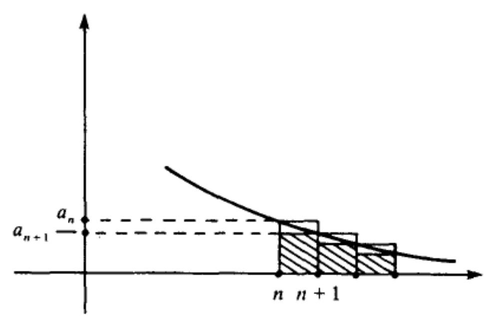

Fig. 37-1

37.65 Estimate the error when $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2}$ is approximated by the first 10 terms.

▌ By Problem 37.64, $R_{10} < \displaystyle\int_{10}^{\infty} \frac{dx}{4x^2} = \lim_{u\to+\infty} \left(-\frac{1}{4x} \bigg]_{10}^{u} \right) = \lim_{u\to+\infty} -\left(\frac{1}{4u} - \frac{1}{40} \right) = \frac{1}{40} = 0.025$. In addition, $R_{10} > \displaystyle\int_{11}^{\infty} \frac{dx}{4x^2} = \lim_{u\to+\infty} \left(-\frac{1}{4x} \bigg]_{11}^{u} \right) = \lim_{u\to+\infty} -\left(\frac{1}{4u} - \frac{1}{44} \right) = \frac{1}{44} \approx 0.0227$. Hence, the error lies between 0.023 and 0.025.

37.66 How many terms are necessary to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ correctly to three decimal places?

▌ By Problem 37.64, the error $R_n < \displaystyle\int_n^{\infty} \frac{dx}{x^3} = \lim_{u\to+\infty} \left(\frac{-1}{2x^2} \bigg]_n^u \right) = \lim_{u\to+\infty} -\frac{1}{2}\left(\frac{1}{u^2} - \frac{1}{n^2} \right) = \frac{1}{2n^2}$. To get $1/2n^2 < 0.005 = \frac{1}{200}$, we need $100 < n^2$, $n > 10$. So at least 11 terms are required.

37.67 What is the error if we approximate a convergent geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ by the first n terms $a + ar + \cdots + ar^{n-1}$?

▌ The error is $ar^n + ar^{n+1} + \cdots$, a geometric series with sum $ar^n/(1-r)$.

37.68 If we approximate the geometric series $\sum_{n=0}^{\infty} \frac{1}{5^n} = 1 + \frac{1}{5} + \frac{1}{25} + \cdots$ by means of the first 10 terms, what will the error be?

I By Problem 37.67, the error is $\frac{(\frac{1}{5})^{10}}{1-\frac{1}{5}} = \frac{1}{4} \cdot \frac{1}{5^9}$.

37.69 For the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p > 1)$, show that the error R_n after n terms is less than $\frac{1}{(p-1)n^{p-1}}$.

I By Problem 37.64, $R_n < \int_n^{\infty} \frac{dx}{x^p} = \lim_{u\to+\infty} \int_n^u \frac{dx}{x^p} = \lim_{u\to+\infty} \left\{ \frac{1}{1-p} x^{-p+1} \Big]_n^u \right\} = \lim_{u\to+\infty} \frac{1}{1-p} \left(\frac{1}{u^{p-1}} - \frac{1}{n^{p-1}} \right) = \frac{1}{p-1} \frac{1}{n^{p-1}}$.

37.70 Study the convergence of $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$.

I Use the ratio test. $\lim_{n\to+\infty} |a_{n+1}/a_n| = \lim_{n\to+\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to+\infty} \frac{2}{n+1} = 0$. Therefore, the series is absolutely convergent.

37.71 Determine whether $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$ converges.

I Use the ratio test.

$$\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to+\infty} \frac{(n+1)!/[2(n+1)]!}{n!/(2n)!} = \lim_{n\to+\infty} \frac{n+1}{(2n+1)(2n+2)} = \lim_{n\to+\infty} \frac{1/n + 1/n^2}{(2+1/n)(2+2/n)} = \frac{0}{4} = 0.$$

Hence, the series converges.

37.72 Determine whether $\sum_{n=1}^{\infty} n(\frac{3}{4})^n$ converges.

I Use the ratio test. $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to+\infty} \frac{(n+1)(\frac{3}{4})^{n+1}}{n(\frac{3}{4})^n} = \lim_{n\to+\infty} \frac{n+1}{n} \cdot \frac{3}{4} = \lim_{n\to+\infty} \frac{3}{4}\left(1+\frac{1}{n}\right) = \frac{3}{4} < 1$. Hence, the series converges.

37.73 Is $\sum_{n=1}^{\infty} n^n e^{-n^2}$ convergent?

I Yes. Since $\ln x/x \le 1/e < \frac{1}{2}$, we have $n^n e^{-n^2} = 1/e^{n^2[1-(\ln n)/n]} < 1/e^{n(1-1/2)} = (1/\sqrt{e})^n$, so the given series is dominated by a convergent geometric series.

37.74 Determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(\pi/n)}{n^2}$ converges.

I $\left| \frac{\sin(\pi/n)}{n^2} \right| \le \frac{1}{n^2}$. Hence, the series is absolutely convergent by comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

37.75 Study the convergence of $1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$.

I The series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}$. Note that, for $n \ge 2$, $\frac{1}{n^n} \le \frac{1}{2^n}$. So, by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, the series is absolutely convergent.

37.76 Study the convergence of $1-\frac{2^2+1}{2^3+1}+\frac{3^2+1}{3^3+1}-\frac{4^2+1}{4^3+1}+\cdots$.

I This is the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+1}{n^3+1}$. By the alternating series test, it is convergent. By the limit comparison test with $\Sigma\, 1/n$, $\lim_{n\to+\infty}\frac{(n^2+1)/(n^3+1)}{1/n}=\lim_{n\to+\infty}\frac{n(n^2+1)}{n^3+1}=\lim_{n\to+\infty}\frac{1+1/n^2}{1+1/n^3}=1$. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$ diverges. Hence, the given series is conditionally convergent.

37.77 Prove the following special case of the ratio test: A series of positive terms $\Sigma\, a_n$ is convergent if $\lim_{n\to+\infty} a_{n+1}/a_n = L < 1$.

I Choose r so that $L<r<1$. There exists an integer k such that, if $n\ge k$, then $|a_{n+1}/a_n - L| < r-L$, and, therefore, $a_{n+1}/a_n = |L+(a_{n+1}/a_n - L)| \le L + |a_{n+1}/a_n - L| < L+(r-L)=r$. So, if $n\ge k$, $a_{n+1}/a_n < r$. Hence, $a_{k+1}/a_k<r$, $a_{k+1}<ra_k$; $a_{k+2}/a_{k+1}<r$, $a_{k+2}<ra_{k+1}<r^2a_k;\ldots;$ $a_{k+m}/a_{k+m-1}<r$, $a_{k+m}<r^m a_k$. Hence, by comparison with the convergent geometric series $ra_k+r^2a_k+r^3a_k+\cdots$, the series $\sum_{i=k+1}^{\infty} a_i = a_{k+1}+a_{k+2}+\cdots$ is convergent, and, therefore, the given series is convergent (since it is obtained from a convergent sequence by addition of a finite number of terms).

37.78 Test $\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}$ for convergence.

I Use the limit comparison test with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. $\frac{n/(n+1)(n+2)(n+3)}{1/n^2}=\frac{n^3}{(n+1)(n+2)(n+3)}=\frac{1}{(1+1/n)(1+2/n)(1+3/n)}\to 1$. Hence, the given series converges.

37.79 Test $\sum_{n=1}^{\infty}\frac{n^3}{(\ln 2)^n}$ for convergence.

I Use the ratio test. $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{(n+1)^3/(\ln 2)^{n+1}}{n^3/(\ln 2)^n}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^3\cdot\frac{1}{\ln 2}=\frac{1}{\ln 2}>1$, since $\ln 2<\ln e=1$. Therefore, the series diverges.

37.80 Test $\sum_{n=1}^{\infty}\frac{n^3}{(\ln 3)^n}$ for convergence.

I Use the ratio test. $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e>1$. Therefore, the series converges.

37.81 Test $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ for convergence.

I Use the ratio test. $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e>1$. Therefore, the series diverges.

37.82 Determine the nth term of and test for convergence the series $\frac{1}{4^2}+\frac{1}{7^2}+\frac{1}{10^2}+\frac{1}{13^2}+\cdots$.

I The nth term is $1/(3n+1)^2$. Use the limit comparison test with the convergent p-series $\Sigma\, 1/n^2$. $\lim_{n\to+\infty}\frac{1/(3n+1)^2}{1/n^2}=\lim_{n\to+\infty}\left(\frac{n}{3n+1}\right)^2=\lim_{n\to+\infty}\left(\frac{1}{3+1/n}\right)^2=\frac{1}{9}$. Therefore, the given series converges.

37.83 Determine the nth term of and test for convergence the series $\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5\cdot 6}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots$.

I The nth term is $1/(n+1)(n+2)\cdots(2n)$. The nth term is less than $1/2\cdot 2\cdots 2=1/2^n$. Hence, the series is convergent by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{2^n}$.

37.84 Determine the nth term of and test for convergence the series $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\frac{4}{3\cdot 5}+\frac{5}{4\cdot 6}+\cdots$.

The nth term is $(n+1)/n(n+2)$. Use the limit comparison test with the divergent series $\Sigma\, 1/n$. $\lim_{n\to+\infty}\frac{n+1/n(n+2)}{1/n}=\lim_{n\to+\infty}\frac{n+1}{n+2}=\lim_{n\to+\infty}\frac{1+1/n}{1+2/n}=1$. Hence, the given series diverges.

37.85 Determine the nth term of and test for convergence the series $\frac{1}{2}+\frac{2}{3^2}+\frac{3}{4^3}+\frac{4}{5^4}+\cdots$.

The nth term is $n/(n+1)^n$. Observe that $\frac{n}{(n+1)^n}=\frac{1}{n^{n-1}}<\frac{1}{n^2}$. Hence, the series converges by comparison with the convergent p-series $\Sigma\, 1/n^2$.

37.86 Determine the nth term of and test for convergence the series $1+\frac{3}{5}+\frac{4}{10}+\frac{5}{17}+\cdots$.

The nth term is $(n+1)/(n^2+1)$. Use the limit comparison test with the divergent series $\Sigma\, 1/n$. $\lim_{n\to+\infty}\frac{(n+1)/(n^2+1)}{1/n}=\lim_{n\to+\infty}\frac{n(n+1)}{n^2+1}=\lim_{n\to+\infty}\frac{1+1/n}{1+1/n^2}=\frac{1}{1}=1$. Therefore, the given series is divergent.

37.87 Determine the nth term of and test for convergence the series $\frac{2}{5}+\frac{2\cdot 4}{5\cdot 8}+\frac{2\cdot 4\cdot 6}{5\cdot 8\cdot 11}+\frac{2\cdot 4\cdot 6\cdot 8}{5\cdot 8\cdot 11\cdot 14}+\cdots$.

The nth term is $\frac{2\cdot 4\cdot\ \cdots\ \cdot(2n)}{5\cdot 8\cdot\ \cdots 1\ \cdot(3n+2)}$. The ratio test yields

$$\lim_{n\to+\infty}\frac{2\cdot 4\cdot\ \cdots\ \cdot(2n)(2n+2)/5\cdot 8\cdot\ \cdots\ \cdot(3n+2)(3n+5)}{2\cdot 4\cdot\ \cdots\ \cdot(2n)/5\cdot 8\cdot\ \cdots\ \cdot(3n+2)}=\lim_{n\to+\infty}\frac{2n+2}{3n+5}=\lim_{n\to+\infty}\frac{2+2/n}{3+5/n}=\frac{2}{3}<1$$

Hence, the series converges.

37.88 Determine the nth term of and test for convergence the series $\frac{3}{2}+\frac{5}{10}+\frac{7}{30}+\frac{9}{68}+\cdots$.

The nth term is $(2n+1)/(n^3+n)$. Use the limit comparison test with the convergent p-series $\Sigma\, 1/n^2$. $\lim_{n\to+\infty}\frac{(2n+1)/(n^3+n)}{1/n^2}=\lim_{n\to+\infty}\frac{n^2(2n+1)}{n(n^2+1)}=\lim_{n\to+\infty}\frac{n(2n+1)}{n^2+1}=\lim_{n\to\infty}\frac{2+1/n}{1+1/n^2}=2$. Hence, the series is convergent.

37.89 Determine the nth term of and test for convergence the series $\frac{3}{2}+\frac{5}{24}+\frac{7}{108}+\frac{9}{320}+\cdots$.

The nth term is $(2n+1)/(n+1)n^3$. Use the limit comparison test with the convergent p-series $\Sigma\, 1/n^3$. $\lim_{n\to+\infty}\frac{(2n+1)/(n+1)n^3}{1/n^3}=\lim_{n\to+\infty}\frac{2n+1}{n+1}=\lim_{n\to+\infty}\frac{2+1/n}{1+1/n}=2$. Hence, the given series converges.

37.90 Determine the nth term of and test for convergence the series $\frac{1}{2^2-1}+\frac{2}{3^2-2}+\frac{3}{4^2-3}+\frac{4}{5^2-4}+\cdots$.

The nth term is $n/[(n+1)^2-n]$. Use the limit comparison test with the divergent series $\Sigma\, 1/n$. $\lim_{n\to+\infty}\frac{n/[(n+1)^2-n]}{1/n}=\lim_{n\to+\infty}\frac{n^2}{n^2+n+1}=\lim_{n\to+\infty}\frac{1}{1+1/n+1/n^2}=1$. Hence, the given series is divergent.

37.91 Prove the *root test*: A series of positive terms $\Sigma\, a_n$ converges if $\lim_{n\to+\infty}\sqrt[n]{a_n}<1$ and diverges if $\lim_{n\to+\infty}\sqrt[n]{a_n}>1$.

Assume $\lim_{n\to+\infty}\sqrt[n]{a_n}=L<1$. Choose r so that $L<r<1$. Then there exists an integer k such that, if $n\ge k$, $\sqrt[n]{a_n}<r$, and, therefore, $a_n<r^n$. Hence, the series $a_k+a_{k+1}+\cdots$ is convergent by comparison with the convergent geometric series $\sum^{\infty} r^n$. So the given series is convergent. Assume now that $\lim_{n\to+\infty}\sqrt[n]{a_n}=L>1$. Choose r so that $L>r>1$. Then there exists an integer k such that, if $n\ge k$, $\sqrt[n]{a_n}>r$, and, therefore, $a_n>r^n$. Thus, by comparison with the divergent geometric series $\sum_{n=k}^{\infty} r^n$, the series $a_k+a_{k+1}+\cdots$ is divergent, and, therefore, the given series is divergent.

37.92. Test $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ for convergence.

Use the root test (Problem 37.91). $\lim \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{1}{\ln n} = 0$. Therefore, the series converges.

37.93 Test $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ for convergence.

Use the root test (Problem 37.91). $\lim \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{2(2^n/n^n)} = \lim_{n\to+\infty} \sqrt[n]{2}(2/n) = 1 \cdot 0 = 0$. Therefore, the series converges.

37.94 Test $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ for convergence.

The root test gives no information, since $\lim_{n\to+\infty} \sqrt[n]{a_n} = 1$. However, $\lim_{n\to+\infty} a_n = e \neq 0$. Hence, the series diverges, by Problem 37.1.

37.95 Determine the nth term of and test for convergence the series $\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{2^3} + \frac{3}{4}\cdot\frac{1}{3^3} - \frac{4}{5}\cdot\frac{1}{4^3} + \cdots$.

$a_n = (-1)^{n+1}[n/(n+1)]/(1/n^3) = (-1)^{n+1}[1/n^2(n+1)]$. Since $|a_n| = 1/n^2(n+1) < 1/n^3$, the given series is absolutely convergent by comparison with the convergent p-series $\Sigma\, 1/n^3$.

37.96 Determine the nth term of and test for convergence the series $\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \frac{5}{6}\cdot\frac{1}{4} + \cdots$.

$a_n = (-1)^{n+1}[(n+1)/(n+2)](1/n)$. The alternating series test implies that the series is convergent. However, it is only conditionally convergent. By the limit comparison test with $\Sigma\, 1/n$, $\lim_{n\to+\infty} \frac{(n+1)/(n+2)n}{1/n} = \lim_{n\to+\infty} \frac{n+1}{n+2} = \lim_{n\to+\infty} \frac{1+1/n}{1+2/n} = 1$. Hence, $\Sigma\, |a_n|$ diverges.

37.97 Determine the nth term of and test for convergence the series $2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots$.

$a_n = 2^{2n-1}/(2n-1)!$. Use the ratio test. $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{2^{2n+1}/(2n+1)!}{2^{2n-1}/(2n-1)!} = \lim_{n\to+\infty} \frac{4}{2n(2n+1)} = 0$. Therefore, the series is absolutely convergent.

37.98 Determine the nth term of and test for convergence the series $\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$.

$a_n = (-1)^{n+1}\, n^2/(n^3+1)$. The series converges by the alternating series test. It is only conditionally convergent, since $\sum |a_n| = \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges, by the limit comparison test with the divergent series $\Sigma\, 1/n$. $\lim_{n\to+\infty} \frac{n^2/(n^3+1)}{1/n} = \lim_{n\to+\infty} \frac{n^3}{n^3+1} = \lim_{n\to+\infty} \frac{1}{1+1/n^3} = 1$.

37.99 Determine the nth term of and test for convergence the series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$.

$a_n = (-1)^{n+1}(n+1)/n$. This is divergent, since $\lim |a_n| = 1 \neq 0$.

37.100 Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$ for convergence.

This series converges by the alternating series test. However, it is only conditionally convergent, since $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ is divergent. To see this, note that $1/\ln n > 1/n$ and use the comparison test with the divergent series $\Sigma\, 1/n$.

37.101 Determine the nth term of and test for convergence the series $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{20}} + \frac{1}{\sqrt{30}} + \cdots$.

▮ $a_n = (-1)^{n+1}/\sqrt{n(n+1)}$. Convergence follows by the alternating series test. However, applying the limit comparison test with $\Sigma\, 1/n$ we see that $\lim_{n\to+\infty} \frac{1/\sqrt{n(n+1)}}{1/n} = \lim_{n\to+\infty} \frac{n}{\sqrt{n(n+1)}} = \lim_{n\to+\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to+\infty} \sqrt{\frac{1}{1+1/n}} = 1$. Therefore, $\sum \frac{1}{\sqrt{n(n+1)}}$ diverges, and the given series is conditionally convergent.

37.102 Determine the nth term of and test for convergence the series $1 - \frac{1}{2^3} + \frac{1}{6^3} - \frac{1}{24^3} + \frac{1}{120^3} + \cdots$.

▮ $a_n = (-1)^{n+1}/(n!)^3$. Use the ratio test $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{1/[(n+1)!]^3}{1/(n!)^3} = \lim_{n\to+\infty} \frac{1}{(n+1)^3} = 0$. Hence, the series is absolutely convergent.

37.103 Show by example that the sum of two divergent series can be convergent.

▮ One trivial example is $\Sigma\, 1/n + \Sigma\,(-1/n) = 0$. Another example is $\Sigma\, 1/n + \Sigma\,(1-n)/n^2 = \Sigma\, 1/n^2$. Of course, the sum of two divergent series of nonnegative terms must be divergent.

37.104 Show how to rearrange the terms of the conditionally convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ so as to obtain a series whose sum is 1.

▮ Use the first n_1 positive terms until the sum is >1. Then use the first n_2 negative terms until the sum becomes <1. Then repeat with more positive terms until the sum becomes >1, then more negative terms until the sum becomes <1, etc. Since the difference between the partial sums and 1 is less than the last term used, the new series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \cdots$ converges to 1. (Note that the series of positive terms $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ and the series of negative terms $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ are both divergent, so the described procedure always can be carried out.)

37.105 Test $\sum_{n=1}^{\infty} \frac{5^n}{n^{n+1}}$ for convergence.

▮ Use the root test. $\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n}} \cdot \frac{5}{n} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n}} \cdot \lim_{n\to+\infty} \frac{5}{n} = 1 \cdot 0 = 0$. (We know that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$ by Problem 36.15.) Hence, the series converges.

37.106 Show that the root test gives no information when $\lim_{n\to+\infty} \sqrt[n]{a_n} = 1$.

▮ Let $a_n = 1/n$. $\Sigma\, 1/n$ is divergent and $\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} 1/\sqrt[n]{n} = 1$. On the other hand, let $a_n = 1/n^2$. Then $\Sigma\, 1/n^2$ is convergent and $\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} 1/(\sqrt[n]{n})^2 = 1$.

37.107 Show that the ratio test gives no information when $\lim |a_{n+1}/a_n| = 1$.

▮ Let $a_n = 1/n$. Then $\Sigma\, 1/n$ is divergent, but $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to+\infty} \frac{n}{n+1} = \lim_{n\to+\infty} \frac{1}{1+1/n} = 1$. On the other hand, let $a_n = 1/n^2$. Then $\Sigma\, 1/n^2$ converges, but $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^2 = \lim_{n\to+\infty} \left(\frac{1}{1+1/n}\right)^2 = 1$.

37.108 Determine whether $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$ converges.

▮ Use the limit comparison test with $\Sigma\, 1/n^{3/2}$. $\lim_{n\to+\infty} \frac{(\ln n)^2/n^2}{1/n^{3/2}} = \lim_{n\to+\infty} \frac{(\ln n)^2}{n^{1/2}} = \lim_{n\to+\infty} \frac{2\ln n \cdot (1/n)}{\frac{1}{2}(1/\sqrt{n})} = \lim_{n\to+\infty} 4\frac{\ln n}{n^{1/2}} = 4 \lim_{n\to+\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = 8 \lim_{n\to+\infty} \frac{1}{n^{1/2}} = 0$. (We have used L'Hôpital's rule twice.) Since $\Sigma\, 1/n^{3/2}$ converges, so does the given series.

37.109 Determine whether $\sum_{n=1}^{\infty} \frac{(-3)^n(1+n^2)}{n!}$ converges.

Use the ratio test. $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{3^{n+1}[1+(n+1)^2]/(n+1)!}{3^n(1+n^2)/n!} = \lim_{n\to+\infty} \frac{3}{n+1}\,\frac{n^2+2n+2}{1+n^2} = \lim_{n\to+\infty} \frac{3}{n+1}\,\frac{1+2/n+2/n^2}{1/n^2+1} = 0\cdot 1 = 0$. Hence, the series is absolutely convergent.

37.110 Show that, in Fig. 37-2, the areas in the rectangles and above $y = 1/x$ add up to a number γ between $\frac{1}{2}$ and 1. (γ is called *Euler's constant.*)

The area in question is less than the sum S of the indicated rectangles. $S = \frac{1}{2} + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots = 1$. So the area is finite and <1. On the other hand, the area is greater than the sum of the triangles (half the rectangles), which is $\frac{1}{2}$. Note that $\gamma = \lim_{n\to+\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right)$. It is an unsolved problem as to whether γ is rational.

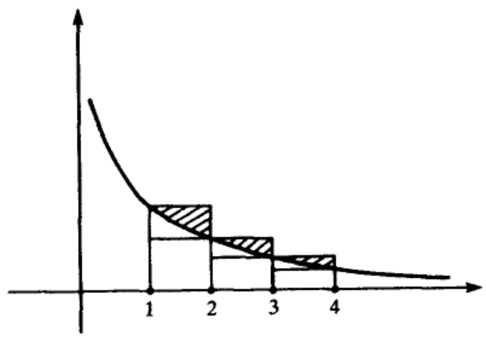

Fig. 37-2

37.111 If $\Sigma\, a_n$ is divergent and $\Sigma\, b_n$ is convergent, show that $\Sigma\,(a_n - b_n)$ is divergent.

Assume $\Sigma\,(a_n - b_n)$ is convergent. Then, $\Sigma\, a_n = \Sigma\, b_n + \Sigma\,(a_n - b_n)$ is convergent, contrary to hypothesis.

37.112 Determine whether $\sum_{n=1}^{\infty}\left[\frac{1}{n} - \left(\frac{2}{3}\right)^n\right]$ converges.

The given series is the difference of a divergent and a convergent series, and is, therefore, by Problem 37.111, divergent.

37.113 Find the values of x for which the series $1 + x + x^2 + \cdots$ converges, and express the sum as a function of x.

This is a geometric series with ratio x. Therefore, it converges for $|x| < 1$. The sum is $1/(1-x)$. Thus, for $|x|<1$, $1/(1-x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$.

37.114 Find the values of x for which the series $x + x^3 + x^5 + \cdots$ converges, and express the sum as a function of x.

This is a geometric series with ratio x^2. Hence, it converges for $|x^2| < 1$, that is, for $|x| < 1$. By the formula $a/(1-r)$ for the sum of a geometric series, the sum is $x/(1-x^2)$.

37.115 Find the values of x for which the series $1/x + 1/x^2 + 1/x^3 + \cdots$ converges and express the sum as a function of x.

This is a geometric series with ratio $1/x$. It converges for $|1/x| < 1$, that is, for $|x| > 1$. The sum is $\frac{1/x}{1-1/x} = \frac{1}{x-1}$.

37.116 Find the values of x for which the series $\ln x + (\ln x)^2 + (\ln x)^3 + \cdots$ converges and express the sum as a function of x.

This is a geometric series with ratio $\ln x$. It converges for $|\ln x| < 1$, $-1 < \ln x < 1$, $1/e < x < e$. The sum is $(\ln x)/(1 - \ln x)$.