Sequences and Series

Find an expression for the nth term of each sequence.

1. $4, 8, 16, 32, 64, \ldots$ exponential growth

$y = 4 \cdot 2^x$

$\boxed{a_n = 4 \cdot 2^{n-1}}$

2. $1, 3, 7, 15, 31, \ldots$ (differences: 2 4 8 16)

$2^n \to 2, 4, 8, 16, 32$

$\therefore \boxed{2^n - 1}$

Not exponential, not linear, so look for patterns. Since, Δa is exponential, a_n is actually exponential too

3. $2, 5, 10, 17, 26, \ldots$ (differences: 3 5 7 9; second differences: 2 2 2)

Not exp, not linear. Δa is linear $\to$ 2^{nd} Δ is constant $\to$ a_n is quadratic

$n^2 \to 1, 4, 9, 16$

$\therefore a_n = \boxed{n^2 + 1}$

4. $1, -3, 5, -7, 9, \ldots$

Linear, but alternates $+/-$. Don't worry about $+/-$ yet.

$1, 3, 5, 7, 9$

$y = 2x + 1$

$a_n = 2(n-1) + 1$

$a_n = \boxed{(-1)^{n+1}(2n-1)}$

$(-1)^{n+1}$ takes care of $+/-$

5. $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$ Do the top & bottom separately

$a_n = \frac{n}{2(n-1)+3}$

$\boxed{\frac{n}{2n+1}}$

6. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{8}, \frac{1}{10}, \ldots$

$a_n = (-1)^{n+1} \frac{1}{2(n-1)+2}$

$a_n = \boxed{\frac{(-1)^{n+1}}{2n}}$

Determine if the **sequence** converges or diverges. If a sequence converges, then find its limit.

7. $(0.2)^n$ $\lim_{n\to\infty} (.2)^n = 0$

$\therefore$ converges to 0

8. 2^n $\lim_{n\to\infty} 2^n$ undefined

$\therefore$ diverges

9. $(-0.3)^n$ $\lim_{n\to\infty} (-.3)^n = 0$

$\therefore$ converges to 0

10. $3 + e^{-2n}$ $\lim_{n\to\infty} (3 + e^{-2n}) = 3 + 0 = 3$

$\therefore$ converges to 3

11. $\frac{2^n}{3^n} \to \left(\frac{2}{3}\right)^n$

$\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$ $\therefore$ conv to 0

12. $\frac{n}{10} + \frac{10}{n}$ $\lim_{n\to\infty} \left(\frac{n}{10} + \frac{10}{n}\right) = \infty + 0$

$\therefore$ diverges

13. $\frac{(-1)^n}{n}$ $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ $\therefore$ conv to 0

the sequence alternates

14. $\frac{2n+1}{n}$ $\lim_{n\to\infty} \frac{2n+1}{n} = 2$

$\therefore$ conv to 2

Find recursive expression for the nth term of each sequence.

15. $\overset{+2}{1}, 3, 5, 7, 9, \ldots$

$a_n = a_{n-1} + 2$,
where $n > 1$
and $a_1 = 1$

16. $3, 5, 9, 17, 33, \ldots$

Look at the pattern of the previous values.
$\therefore$ twice the previous -1
$2a_{n-1} - 1$, where
$n > 1$ and $a_1 = 3$

The nth Term Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ is a divergent series.

Note: the test **does not** conclude that a series converges. If the limit is any other number besides 0, then we can only conclude that the series is divergent. If the limit is 0, then there is no conclusion. If the limit is 0, the series is probably convergent, but there are many exceptions such as $1/x$.

Use the nth Term Divergence Test to determine if the **series** converges or diverges.

17. $\sum\limits_{n=1}^{\infty} \frac{n+1}{n}$

$\lim\limits_{n\to\infty} a_n = 1 \neq 0$
$\therefore$ diverges

18. $\sum\limits_{n=1}^{\infty} \frac{n^2+n}{n^3}$

$\lim\limits_{n\to\infty} a_n = 0$ $\therefore$ No Concl.

19. $\sum\limits_{n=1}^{\infty} \frac{1+3n^2+n^3}{4n^3-5n+2}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{4} \neq 0$
$\therefore$ diverges

20. $\sum\limits_{n=1}^{\infty} \cos(\pi n)$

$\lim\limits_{n\to\infty} a_n$ undefined $\neq 0$
$\therefore$ diverges

21. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^2}$

$\lim\limits_{n\to\infty} \frac{\ln n}{n^2} = 0$
$\therefore$ No Concl.

22. $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n}$

$\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$
$\therefore$ No Concl.

23. $\sum\limits_{n=1}^{\infty} \frac{2n+(-1)^n 5}{4n-(-1)^n 3}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0$ $\therefore$ diverges

24. $\sum\limits_{n=1}^{\infty} \frac{2^n}{n^3}$

$\lim\limits_{n\to\infty} a^n$ undefined $\neq 0$
$\therefore$ diverges

Evaluate the sum of each **telescoping series**, and find an expression for the ***n* th Partial Sum** of the series.

25. $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$

$S_1 = \frac{1}{1} - \frac{1}{3}$

$S_2 = \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{4}\right]$

$\vdots$

remains

$S_6 = \left[\frac{1}{1} - \cancel{\frac{1}{3}}\right] + \left[\frac{1}{2} - \cancel{\frac{1}{4}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right] + \left[\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right] + \left[\cancel{\frac{1}{5}} - \frac{1}{7}\right] + \left[\cancel{\frac{1}{6}} - \frac{1}{8}\right]$

remains

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

$S_n = \frac{3(n+1)(n+2)}{2(n+1)(n+2)} - \frac{2(n+2)}{2(n+1)(n+2)} - \frac{2(n+1)}{2(n+1)(n+2)} \rightarrow \frac{3(n^2+3n+2)-2n-4-2n-2}{2(n^2+3n+2)} \rightarrow \boxed{\frac{3n^2+5n}{2n^2+6n+4}}$

26. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ (Hint: rewrite using partial fractions first)

$(n+3)(n+1)$

$\frac{2}{n^2+4n+3} = \frac{A}{n+3} + \frac{B}{n+1}$

$2 = A(n+1) + B(n+3)$

If $n = -1$, $2 = B(2) \rightarrow B = 1$

If $n = -3$, $2 = A(-2) \rightarrow A = -1$

$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \boxed{\sum_{n=1}^{\infty} \frac{-1}{n+3} + \frac{1}{n+1}}$ ← Use this for S_6.

$S_6 = \left[-\cancel{\frac{1}{4}} + \frac{1}{2}\right] + \left[-\cancel{\frac{1}{5}} + \frac{1}{3}\right] + \left[-\cancel{\frac{1}{6}} + \cancel{\frac{1}{4}}\right] + \left[-\cancel{\frac{1}{7}} + \cancel{\frac{1}{5}}\right] + \left[-\frac{1}{8} + \cancel{\frac{1}{6}}\right] + \left[-\frac{1}{9} + \cancel{\frac{1}{7}}\right]$

$\therefore \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$

$\boxed{S_n = \frac{5}{6} + -\frac{1}{n+2} - \frac{1}{n+3}} \rightarrow \frac{5(n^2+5n+6)}{6(n+2)(n+3)} + \frac{-6(n+3)}{6(n+2)(n+3)} + \frac{-6(n+2)}{6(n+2)(n+3)} \rightarrow \boxed{\frac{5n^2+13n}{6n^2+30n+36}}$

27. The infinite series $\sum_{k=1}^{\infty} a_k$ has nth partial sum $S_n = \frac{n}{3n+1}$ for $n \geq 1$. What is the sum of the series $\sum_{k=1}^{\infty} a_k$?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) The series diverges.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$$

$$= \boxed{\frac{1}{3}}$$

28. The infinite series $\sum_{k=1}^{\infty} a_k$ has nth partial sum $S_n = (-1)^{n+1}$ for $n \geq 1$. What is the sum of the series $\sum_{k=1}^{\infty} a_k$?

(A) -1 (B) 0 (C) $\frac{1}{2}$ (D) 1 (E) The series diverges.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$$

the sequence $(-1)^{n+1}$ alternates from $+, -, +, -, \ldots$

$\neq 0$