Change of Variable For Definite Integral
Section 6.7

Note. Let us go back to the example $\int_{-2}^{4} \sqrt{x+4}\,dx$ we worked on in the last lecture. There we made the change of variable $u = x + 4$. Using this change of variable , we changed the associated indefinite integral to the indefinite integral $\frac{2}{3}u^{\frac{3}{2}} + C$ and then change it to something in terms of x. Another option is this:

First: find the associated values for u for the upper and lower limits of the integral $\int_{-2}^{4} \sqrt{x+4}\,dx$ which are

$$\begin{cases} x = 4 & \overset{u=x+4}{\Rightarrow} \quad u = 8 \\ x = -2 & \overset{u=x+4}{\Rightarrow} \quad u = 2 \end{cases}$$

Second: continue in this way:

$\int_{-2}^{4} \sqrt{x+4}\,dx = \int_{2}^{8} \sqrt{u}du = \int_{2}^{8} u^{\frac{1}{2}}du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=2}^{u=8} = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{u=2}^{u=8}$

$= \frac{2}{3}\left[16\sqrt{2} - 2\sqrt{2}\right] = \frac{28\sqrt{2}}{3}$

which is the same answer as above. In this method we do not calculate the indefinite integral involving x . This is the whole story portrayed by Theorem 6.9 on page 405.

Example (section 6.7 exercise 15). Evaluate the integral

$$\int_4^9 \sqrt{1+\sqrt{x}}\,dx$$

Solution.

$$u = 1+\sqrt{x} = 1+x^{\frac{1}{2}} \quad \Rightarrow \quad du = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx = \tfrac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow \quad dx = 2\sqrt{x}\,du = 2(u-1)du$$

Next Step.

$$\begin{cases} x=9 \;\Rightarrow\; u=4 \\ x=4 \;\Rightarrow\; u=3 \end{cases}$$

Next Step.

$$\int_4^9 \sqrt{1+\sqrt{x}}\,dx = \int_3^4 2\sqrt{u}(u-1)\,du = 2\int_3^4 u^{\frac{1}{2}}(u-1)\,du$$

$$= 2\int_3^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = 2\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right]_{u=3}^{u=4} = 2\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right]_{u=3}^{u=4}$$

$$= 2\left\{\left(\tfrac{64}{5} - \tfrac{16}{3}\right) - \left(\tfrac{18\sqrt{3}}{5} - 2\sqrt{3}\right)\right\} = \tfrac{224-48\sqrt{3}}{15}$$

Theorem. For three points a and b and c the following equality holds provided that the three integrals exist:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Note. We will use the above theorem in the next example to evaluate the given integral:

Example (section 6.7 exercise 17). Evaluate the integral

$$\int_{-1}^{1} \frac{|x|}{(x+2)^3}\,dx$$

Solution.

$$\int_{-1}^{1} \frac{|x|}{(x+2)^3}\,dx = \underbrace{\int_{-1}^{0} \frac{(-x)}{(x+2)^3}\,dx}_{\mathbf{A}} + \underbrace{\int_{0}^{1} \frac{x}{(x+2)^3}\,dx}_{\mathbf{B}}$$

Next Step.

$$A = -\int_{-1}^{0} \frac{x}{(x+2)^3}\,dx \qquad \begin{cases} u = x+2 \\ du = dx \end{cases} \qquad \begin{cases} x = 0 \;\Rightarrow\; u = 2 \\ x = -1 \;\Rightarrow\; u = 1 \end{cases}$$

$$= -\int_{1}^{2} \frac{u-2}{u^3}\,du = -\int_{1}^{2} (u-2)u^{-3}\,du = -\int_{1}^{2} \left(u^{-2} - 2u^{-3}\right)\,du$$

$$= -\left[\frac{u^{-1}}{-1} - 2\left(\frac{u^{-2}}{-2}\right)\right]_1^2 = -\left[-\frac{1}{u} + \frac{1}{u^2}\right]_1^2 = \left[\frac{1}{u} - \frac{1}{u^2}\right]_1^2$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{4}\right) - (1-1) = \tfrac{1}{4}$$

Next Step.

$$\mathbf{B} = \int_0^1 \frac{x}{(x+2)^3}\,dx \qquad \begin{cases} u = x+2 \\ du = dx \end{cases} \qquad \begin{cases} x = 1 \;\Rightarrow\; u = 3 \\ x = 0 \;\Rightarrow\; u = 2 \end{cases}$$

$$= \int_2^3 \frac{u-2}{u^3}\,du = \int_2^3 (u-2)u^{-3}\,du = \int_2^3 \left(u^{-2} - 2u^{-3}\right)\,du$$

$$= \left[\frac{u^{-1}}{-1} - 2\left(\frac{u^{-2}}{-2}\right)\right]_2^3 = \left[-\frac{1}{u} + \frac{1}{u^2}\right]_2^3 = \left(-\frac{1}{3} + \frac{1}{9}\right) - \left(-\frac{1}{2} + \frac{1}{4}\right) = \frac{1}{36}$$

Next Step.

$$\int_{-1}^1 \frac{|x|}{(x+2)^3}\,dx = \mathbf{A} + \mathbf{B} = \frac{1}{4} + \frac{1}{36} = \frac{5}{18}$$

Example (section 6.7 exercise 19). Evaluate the integral $\int_0^1 x^2\, e^{x^3}\, dx$

Solution.

$u = x^3 \quad \Rightarrow \quad du = 3x^2\, dx \quad \Rightarrow \quad x^2 dx = \frac{1}{3} du \quad \Rightarrow \quad x^2 e^{x^3} dx = \frac{1}{3} e^u\, du$

or do this:

$u = x^3 \quad \Rightarrow \quad du = 3x^2\, dx \quad \Rightarrow \quad dx = \frac{1}{3x^2} du = \frac{1}{3(\sqrt[3]{u})^2}\, du$

$\Rightarrow \quad x^2\, e^{x^3}\, dx = (\sqrt[3]{u})^2 e^u \frac{1}{3(\sqrt[3]{u})^2}\, du = \frac{1}{3} e^u\, du$

Next Step.

$$\begin{cases} x = 1 \;\Rightarrow\; u = 1 \\ x = 0 \;\Rightarrow\; u = 0 \end{cases}$$

Next Step.

$$\int_{x=0}^{x=1} x^2\, e^{x^3}\, dx = \int_{u=0}^{u=1} \frac{1}{3}\, e^u\, du = \frac{1}{3} \int_{u=0}^{u=1} e^u\, du$$

$$= \frac{1}{3} \Big[e^u \Big]_{u=0}^{u=1} = \frac{1}{3}(e^1 - e^0) = \frac{e-1}{3}$$

Example (section 6.7 exercise 8). Evaluate the integral

$$\int_{1/2}^{1} \sqrt{\frac{x^2}{1+x}}\, dx$$

Solution.

Write the integral in the form

$$\int_{1/2}^{1} \frac{x}{\sqrt{1+x}}\, dx$$

and then make the following change of variable:

$$\begin{cases} u = 1 + x \\ du = dx \end{cases} \qquad \begin{cases} x = 1 \;\Rightarrow\; u = 2 \\ x = \frac{1}{2} \;\Rightarrow\; u = \frac{3}{2} \end{cases}$$

Next Step.

$$\int_{1/2}^{1} \frac{x}{\sqrt{1+x}}\, dx = \int_{3/2}^{2} \frac{u-1}{\sqrt{u}}\, du = \int_{3/2}^{2} \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right)\, du$$

$$= \int_{3/2}^{2} \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right)\, du = \int_{3/2}^{2} \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right)\, du = \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_{3/2}^{2}$$

$$= \left[\frac{2}{3} u\sqrt{u} - 2\sqrt{u} \right]_{3/2}^{2} = \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} \right) - \left(\sqrt{\frac{3}{2}} - 2\sqrt{\frac{3}{2}} \right) = \sqrt{\frac{3}{2}} - \frac{2\sqrt{2}}{3}$$

Important Note. Sometimes in your work you may face integrals $\int_a^b f(x)\,dx$ where the upper boundary b is actually smaller than the lower boundary a. However , in these cases too the First Fundamental Theorem of Calculus will be applied as usual; see the following example for this case:

Example . Evaluate the integral

$$\int_0^{\frac{1}{2}} \frac{x^3}{1-x^4}\,dx$$

Solution.

$$\begin{cases} u = 1 - x^4 \\ du = -4x^3\,dx \end{cases} \qquad \begin{cases} x = \frac{1}{2} \Rightarrow u = \frac{15}{16} \\ x = 0 \Rightarrow u = 1 \end{cases}$$

Then

$$\int_0^{\frac{1}{2}} \frac{x^3}{1-x^4}\,dx = \int_1^{\frac{15}{16}} \frac{-\frac{1}{4}\,du}{u} = -\frac{1}{4}\int_1^{\frac{15}{16}} \frac{1}{u}\,du = -\frac{1}{4}\Big[\ln|u|\Big]_{u=1}^{u=\frac{15}{16}}$$

$$= -\tfrac{1}{4}\Big\{\ln(\tfrac{15}{16}) - \ln(1)\Big\} = -\tfrac{1}{4}\Big\{\ln(\tfrac{15}{16}) - 0\Big\} = -\tfrac{\ln(\frac{15}{16})}{4}$$