Integration by Substitution

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use the General Power Rule for Integration to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.
- Evaluate a definite integral involving an even or odd function.

Pattern Recognition

In this section, you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables.* Both techniques involve a ***u*-substitution.** With pattern recognition, you perform the substitution mentally, and with change of variables, you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. Recall that for the differentiable functions

$$y = F(u) \quad \text{and} \quad u = g(x)$$

the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the next theorem.

THEOREM 4.13 Antidifferentiation of a Composite Function

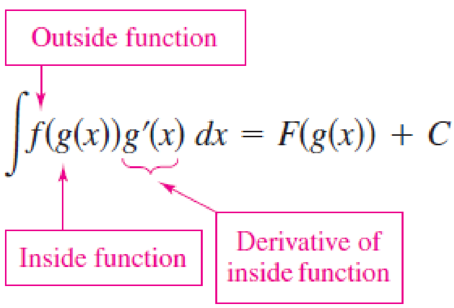
Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Examples 1 and 2 show how to apply Theorem 4.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$. Note that the composite function in the integrand has an *outside function* f and an *inside function* g. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.



EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

EXAMPLE 2 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int 5 \cos 5x \, dx$.

Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Cosine Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(\cos 5x)}^{f(g(x))}\overbrace{(5)}^{g'(x)} \, dx = \sin 5x + C.$$

You can check this by differentiating $\sin 5x + C$ to obtain the original integrand.

Exploration

Recognizing Patterns The integrand in each of the integrals labeled (a)–(c) fits the pattern $f(g(x))g'(x)$. Identify the pattern and use the result to evaluate the integral.

a. $\int 2x(x^2 + 1)^4\,dx$ **b.** $\int 3x^2\sqrt{x^3 + 1}\,dx$ **c.** $\int \sec^2 x(\tan x + 3)\,dx$

The integrals labeled (d)–(f) are similar to (a)–(c). Show how you can multiply and divide by a constant to evaluate these integrals.

d. $\int x(x^2 + 1)^4\,dx$ **e.** $\int x^2\sqrt{x^3 + 1}\,dx$ **f.** $\int 2\sec^2 x(\tan x + 3)\,dx$

The integrands in Examples 1 and 2 fit the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern. You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

Find the indefinite integral.

$$\int x(x^2 + 1)^2 \, dx$$

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let

$$g(x) = x^2 + 1$$

and supply the $2x$ as shown.

$$\int x(x^2 + 1)^2 \, dx = \int (x^2 + 1)^2 \left(\frac{1}{2}\right)(2x) \, dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))} \overbrace{(2x)}^{g'(x)} \, dx \qquad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C \qquad \text{Integrate.}$$

$$= \frac{1}{6}(x^2 + 1)^3 + C \qquad \text{Simplify.}$$

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\int x(x^2 + 1)^2\,dx = \frac{1}{2}\int (x^2 + 1)^2\,(2x)\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C$$

$$= \frac{1}{6}(x^2 + 1)^3 + C.$$

Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You cannot multiply and divide by a variable and then move the variable outside the integral sign. For instance,

$$\int (x^2 + 1)^2\,dx \neq \frac{1}{2x}\int (x^2 + 1)^2\,(2x)\,dx.$$

After all, if it were legitimate to move variable quantities outside the integral sign, you could move the entire integrand out and simplify the whole process. But the result would be incorrect.

Change of Variables

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

EXAMPLE 4 Change of Variables

Find $\int \sqrt{2x - 1}\, dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\, dx$. Now, using $\sqrt{2x - 1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\int \sqrt{2x - 1}\, dx = \int \sqrt{u}\left(\frac{du}{2}\right) \qquad \text{Integral in terms of } u$$

$$= \frac{1}{2}\int u^{1/2}\, du \qquad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C \qquad \text{Antiderivative in terms of } u$$

$$= \frac{1}{3}u^{3/2} + C \qquad \text{Simplify.}$$

$$= \frac{1}{3}(2x - 1)^{3/2} + C. \qquad \text{Antiderivative in terms of } x$$

EXAMPLE 5 Change of Variables

See LarsonCalculus.com for an interactive version of this type of example.

Find $\int x\sqrt{2x-1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$$u = 2x - 1 \quad \Rightarrow \quad x = \frac{u+1}{2} \qquad \text{Solve for } x \text{ in terms of } u.$$

Now, using substitution, you obtain

$$\begin{aligned}
\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right) u^{1/2} \left(\frac{du}{2}\right) \\
&= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du \\
&= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C \\
&= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\end{aligned}$$

To complete the change of variables in Example 5, you solved for x in terms of u. Sometimes this is very difficult. Fortunately, it is not always necessary, as shown in the next example.

EXAMPLE 6 Change of Variables

Find $\int \sin^2 3x \cos 3x \, dx$.

Solution Because $\sin^2 3x = (\sin 3x)^2$, you can let $u = \sin 3x$. Then

$$du = (\cos 3x)(3)\, dx.$$

Now, because $\cos 3x \, dx$ is part of the original integral, you can write

$$\frac{du}{3} = \cos 3x \, dx.$$

Substituting u and $du/3$ in the original integral yields

$$\begin{aligned}
\int \sin^2 3x \cos 3x \, dx &= \int u^2 \frac{du}{3} \\
&= \frac{1}{3}\int u^2 \, du \\
&= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \\
&= \frac{1}{9}\sin^3 3x + C.
\end{aligned}$$

You can check this by differentiating.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{9}\sin^3 3x + C\right] &= \left(\frac{1}{9}\right)(3)(\sin 3x)^2(\cos 3x)(3) \\
&= \sin^2 3x \cos 3x
\end{aligned}$$

Because differentiation produces the original integrand, you know that you have obtained the correct antiderivative. ■

The steps used for integration by substitution are summarized in the following guidelines.

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answers by differentiating.

So far, you have seen two techniques for applying substitution, and you will see more techniques in the remainder of this section. Each technique differs slightly from the others. You should remember, however, that the goal is the same with each technique—*you are trying to find an antiderivative of the integrand.*

The General Power Rule for Integration

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.** A proof of this rule follows directly from the (simple) Power Rule for Integration, together with Theorem 4.13.

THEOREM 4.14 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $$\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)}^{du}\,dx = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$$

b. $$\int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$$

c. $$\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$$

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\overbrace{\frac{(\cos x)^3}{3}}^{u^3/3} + C$$ ■

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2 + 1)^2 \, dx \quad \text{and} \quad \int (x^2 + 1)^2 \, dx.$$

The substitution

$$u = x^2 + 1$$

works in the first integral, but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as

$$(x^2 + 1)^2 = x^4 + 2x^2 + 1$$

and use the (simple) Power Rule to integrate each term.

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.13 combined with the Fundamental Theorem of Calculus.

THEOREM 4.15 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.$$

EXAMPLE 8 Change of Variables

Evaluate $\int_0^1 x(x^2 + 1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \quad \Rightarrow \quad du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	**Upper Limit**
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\int_0^1 x(x^2 + 1)^3\,dx = \frac{1}{2}\int_0^1 (x^2 + 1)^3(2x)\,dx \qquad \text{Integration limits for } x$$

$$= \frac{1}{2}\int_1^2 u^3\,du \qquad \text{Integration limits for } u$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}.$$

Notice that you obtain the same result when you rewrite the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown below.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2 + 1)^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}$$

EXAMPLE 9 Change of Variables

Evaluate the definite integral.

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$$

Solution To evaluate this integral, let $u = \sqrt{2x-1}$. Then, you obtain

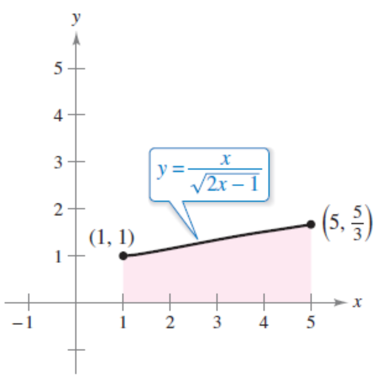
$$u^2 = 2x - 1$$
$$u^2 + 1 = 2x$$
$$\frac{u^2+1}{2} = x$$
$$u\,du = dx. \qquad \text{Differentiate each side.}$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	**Upper Limit**
When $x = 1$, $u = \sqrt{2-1} = 1$.	When $x = 5$, $u = \sqrt{10-1} = 3$.

Now, substitute to obtain

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{1}{u}\left(\frac{u^2+1}{2}\right) u\,du$$
$$= \frac{1}{2}\int_1^3 (u^2+1)\,du$$
$$= \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3$$
$$= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right)$$
$$= \frac{16}{3}.$$



The region before substitution has an area of $\frac{16}{3}$.

Figure 4.38

The region before substitution has an area of $\frac{16}{3}$.
Figure 4.38

Geometrically, you can interpret the equation

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \frac{u^2+1}{2}\,du$$

to mean that the two *different* regions shown in Figures 4.38 and 4.39 have the *same* area.

When evaluating definite integrals by substitution, it is possible for the upper limit of integration of the u-variable form to be smaller than the lower limit. When this happens, don't rearrange the limits. Simply evaluate as usual. For example, after substituting $u = \sqrt{1-x}$ in the integral

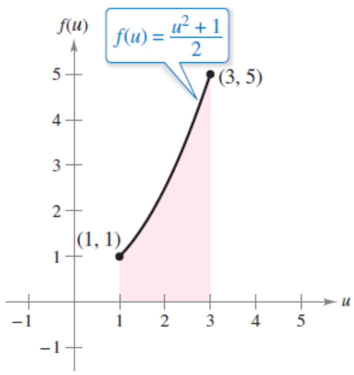
$$\int_0^1 x^2(1-x)^{1/2}\,dx$$

you obtain $u = \sqrt{1-1} = 0$ when $x = 1$, and $u = \sqrt{1-0} = 1$ when $x = 0$. So, the correct u-variable form of this integral is

$$-2\int_1^0 (1-u^2)^2 u^2\,du.$$

Expanding the integrand, you can evaluate this integral as shown.

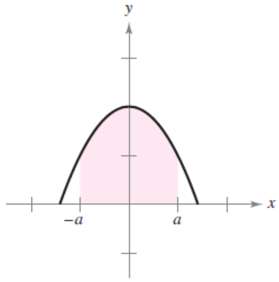
$$-2\int_1^0 (u^2 - 2u^4 + u^6)\,du = -2\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right]_1^0 = -2\left(-\frac{1}{3} + \frac{2}{5} - \frac{1}{7}\right) = \frac{16}{105}$$



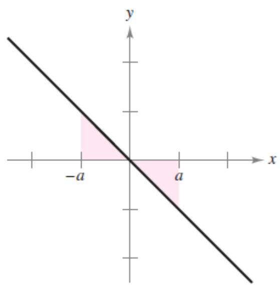
The region after substitution has an area of $\frac{16}{3}$.
Figure 4.39

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult. Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the y-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).



Even function



Odd function

Figure 4.40

> **THEOREM 4.16 Integration of Even and Odd Functions**
>
> Let f be integrable on the closed interval $[a, -a]$.
>
> **1.** If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
>
> **2.** If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$

Proof Here is the proof of the first property. (The proof of the second property is left to you [see Exercise 99].) Because f is even, you know that $f(x) = f(-x)$. Using Theorem 4.13 with the substitution $u = -x$ produces

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Finally, using Theorem 4.6, you obtain

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

EXAMPLE 10 Integration of an Odd Function

Evaluate the definite integral.

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx$$

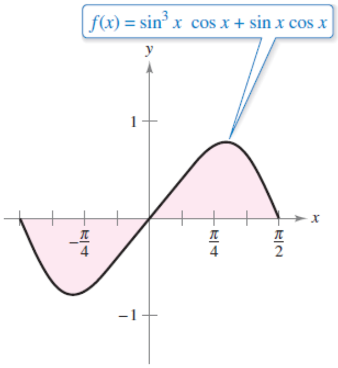
Solution Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$\begin{aligned} f(-x) &= \sin^3(-x)\cos(-x) + \sin(-x)\cos(-x) \\ &= -\sin^3 x \cos x - \sin x \cos x \\ &= -f(x). \end{aligned}$$

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\,dx = 0.$$

From Figure 4.41, you can see that the two regions on either side of the y-axis have the same area. However, because one lies below the x-axis and one lies above it, integration produces a cancellation effect. (More will be said about areas below the x-axis in Section 7.1.)



Because f is an odd function, $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$

Figure 4.41

4.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding u and du In Exercises 1–4, complete the table by identifying u and du for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2+1)^2(16x)\,dx$		
2. $\int x^2\sqrt{x^3+1}\,dx$		
3. $\int \tan^2 x \sec^2 x\,dx$		
4. $\int \frac{\cos x}{\sin^2 x}\,dx$		

Finding an Indefinite Integral In Exercises 5–26, find the indefinite integral and check the result by differentiation.

5. $\int (1+6x)^4(6)\,dx$

6. $\int (x^2-9)^3(2x)\,dx$

7. $\int \sqrt{25-x^2}\,(-2x)\,dx$

8. $\int \sqrt[3]{3-4x^2}(-8x)\,dx$

9. $\int x^3(x^4+3)^2\,dx$

10. $\int x^2(6-x^3)^5\,dx$

11. $\int x^2(x^3-1)^4\,dx$

12. $\int x(5x^2+4)^3\,dx$

13. $\int t\sqrt{t^2+2}\,dt$

14. $\int t^3\sqrt{2t^4+3}\,dt$

15. $\int 5x\,\sqrt[3]{1-x^2}\,dx$

16. $\int u^2\sqrt{u^3+2}\,du$

17. $\int \frac{x}{(1-x^2)^3}\,dx$

18. $\int \frac{x^3}{(1+x^4)^2}\,dx$

19. $\int \frac{x^2}{(1+x^3)^2}\,dx$

20. $\int \frac{6x^2}{(4x^3-9)^3}\,dx$

21. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

22. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

23. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

24. $\int \left[x^2+\frac{1}{(3x)^2}\right]dx$

25. $\int \frac{1}{\sqrt{2x}}\,dx$

26. $\int \frac{x}{\sqrt[3]{5x^2}}\,dx$

Differential Equation In Exercises 27–30, solve the differential equation.

27. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

28. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

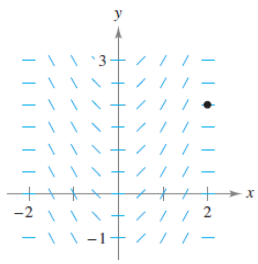
29. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

30. $\frac{dy}{dx} = \frac{x-4}{\sqrt{x^2-8x+1}}$

Slope Field In Exercises 31 and 32, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

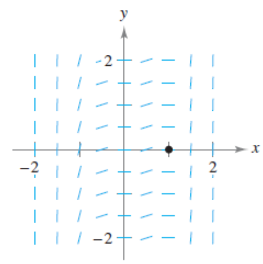
31. $\frac{dy}{dx} = x\sqrt{4-x^2}$

$(2, 2)$



32. $\frac{dy}{dx} = x^2(x^3-1)^2$

$(1, 0)$



Finding an Indefinite Integral In Exercises 33–42, find the indefinite integral.

33. $\int \pi \sin \pi x\,dx$

34. $\int \sin 4x\,dx$

35. $\int \cos 8x\,dx$

36. $\int \csc^2\left(\frac{x}{2}\right)dx$

37. $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$

38. $\int x \sin x^2\,dx$

39. $\int \sin 2x \cos 2x\,dx$

40. $\int \sqrt{\tan x}\,\sec^2 x\,dx$

41. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

42. $\int \frac{\sin x}{\cos^3 x}\,dx$

Finding an Equation In Exercises 43–46, find an equation for the function f that has the given derivative and whose graph passes through the given point.

Derivative	Point
43. $f'(x) = -\sin\frac{x}{2}$	$(0, 6)$
44. $f'(x) = \sec^2(2x)$	$\left(\frac{\pi}{2}, 2\right)$
45. $f'(x) = 2x(4x^2-10)^2$	$(2, 10)$
46. $f'(x) = -2x\sqrt{8-x^2}$	$(2, 7)$

Change of Variables In Exercises 47–54, find the indefinite integral by the method shown in Example 5.

47. $\int x\sqrt{x+6}\,dx$

48. $\int x\sqrt{3x-4}\,dx$

49. $\int x^2\sqrt{1-x}\,dx$

50. $\int (x+1)\sqrt{2-x}\,dx$

51. $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

52. $\int \frac{2x+1}{\sqrt{x+4}}\,dx$

53. $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

54. $\int t\sqrt[3]{t+10}\,dt$

Evaluating a Definite Integral In Exercises 55–62, evaluate the definite integral. Use a graphing utility to verify your result.

55. $\int_{-1}^{1} x(x^2+1)^3\,dx$

56. $\int_{0}^{1} x^3(2x^4+1)^2\,dx$

57. $\int_{1}^{2} 2x^2\sqrt{x^3+1}\,dx$

58. $\int_{0}^{1} x\sqrt{1-x^2}\,dx$

59. $\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\,dx$

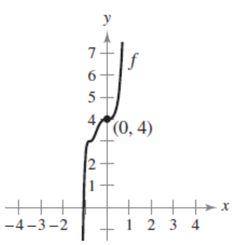
60. $\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\,dx$

61. $\int_{1}^{9} \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

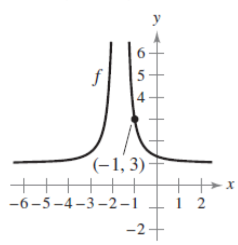
62. $\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$

Differential Equation In Exercises 63 and 64, the graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.

63. $\frac{dy}{dx} = 18x^2(2x^3+1)^2$

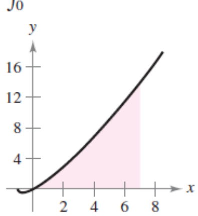


64. $\frac{dy}{dx} = \frac{-48}{(3x+5)^3}$

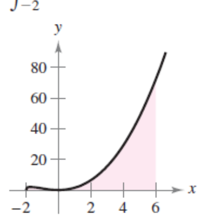


Finding the Area of a Region In Exercises 65–68, find the area of the region. Use a graphing utility to verify your result.

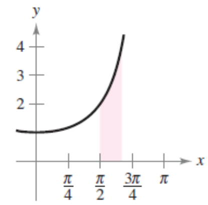
65. $\int_{0}^{7} x\sqrt[3]{x+1}\,dx$



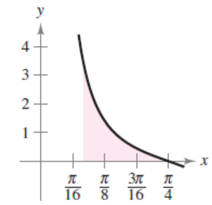
66. $\int_{-2}^{6} x^2\sqrt[3]{x+2}\,dx$



67. $\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx$



68. $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx$



Even and Odd Functions In Exercises 69–72, evaluate the integral using the properties of even and odd functions as an aid.

69. $\int_{-2}^{2} x^2(x^2+1)\,dx$

70. $\int_{-2}^{2} x(x^2+1)^3\,dx$

71. $\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx$

72. $\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx$

73. **Using an Even Function** Use $\int_0^4 x^2\,dx = \frac{64}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\int_{-4}^{0} x^2\,dx$ (b) $\int_{-4}^{4} x^2\,dx$

(c) $\int_{0}^{4} -x^2\,dx$ (d) $\int_{-4}^{0} 3x^2\,dx$

74. **Using Symmetry** Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx$ (b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx$

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$ (d) $\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx$

Even and Odd Functions In Exercises 75 and 76, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

75. $\int_{-3}^{3} (x^3+4x^2-3x-6)\,dx$

76. $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\,dx$

WRITING ABOUT CONCEPTS

77. **Using Substitution** Describe why

$$\int x(5-x^2)^3\,dx \neq \int u^3\,du$$

where $u = 5 - x^2$.

78. **Analyzing the Integrand** Without integrating, explain why

$$\int_{-2}^{2} x(x^2+1)^2\,dx = 0.$$

WRITING ABOUT CONCEPTS (continued)

79. Choosing an Integral You are asked to find one of the integrals. Which one would you choose? Explain.

(a) $\int \sqrt{x^3+1}\,dx$ or $\int x^2\sqrt{x^3+1}\,dx$

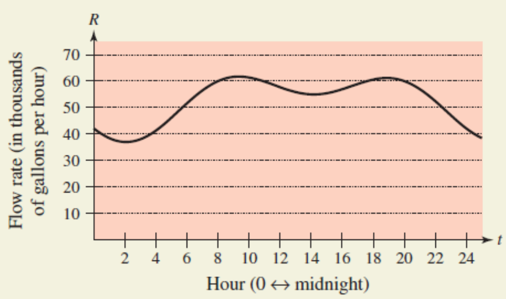
(b) $\int \tan(3x)\sec^2(3x)\,dx$ or $\int \tan(3x)\,dx$

80. Comparing Methods Find the indefinite integral in two ways. Explain any difference in the forms of the answers.

(a) $\int (2x-1)^2\,dx$ (b) $\int \tan x \sec^2 x\,dx$

81. Depreciation The rate of depreciation dV/dt of a machine is inversely proportional to the square of $(t+1)$, where V is the value of the machine t years after it was purchased. The initial value of the machine was $500,000, and its value decreased $100,000 in the first year. Estimate its value after 4 years.

82. HOW DO YOU SEE IT? The graph shows the flow rate of water at a pumping station for one day.



(a) Approximate the maximum flow rate at the pumping station. At what time does this occur?

(b) Explain how you can find the amount of water used during the day.

(c) Approximate the two-hour period when the least amount of water is used. Explain your reasoning.

83. Sales The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

84. Electricity

The oscillating current in an electrical circuit is

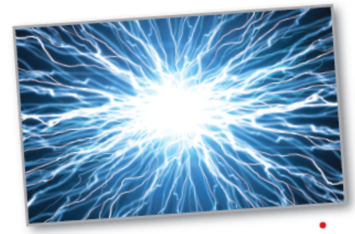
$I = 2\sin(60\pi t) + \cos(120\pi t)$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$



Probability **In Exercises 85 and 86, the function**

$$f(x) = kx^n(1-x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\,dx = 1$$

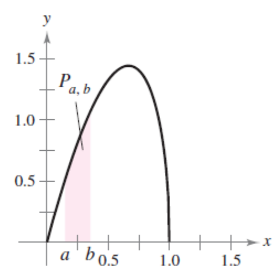
then the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,b} = \int_a^b f(x)\,dx.$$

85. The probability that a person will remember between $100a\%$ and $100b\%$ of material learned in an experiment is

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx$$

where x represents the proportion remembered. (See figure.)



(a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

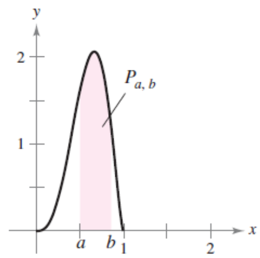
(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

86. The probability that ore samples taken from a region contain between $100a\%$ and $100b\%$ iron is

$$P_{a,\,b} = \int_a^b \frac{1155}{32} x^3(1 - x)^{3/2}\,dx$$

where x represents the proportion of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron? (b) 50% and 100% iron?



87. Graphical Analysis Consider the functions f and g, where

$$f(x) = 6 \sin x \cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function

$$h(t) = \int_{\pi/2}^t f(x)\,dx.$$

Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

88. Finding a Limit Using a Definite Integral Find

$$\lim_{n \to +\infty} \sum_{i=1}^{n} \frac{\sin(i\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval $[0, 1]$.

89. Rewriting Integrals

(a) Show that $\int_0^1 x^2(1 - x)^5\,dx = \int_0^1 x^5(1 - x)^2\,dx$.

(b) Show that $\int_0^1 x^a(1 - x)^b\,dx = \int_0^1 x^b(1 - x)^a\,dx$.

90. Rewriting Integrals

(a) Show that $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx$.

(b) Show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. $\int (2x + 1)^2\,dx = \frac{1}{3}(2x + 1)^3 + C$

92. $\int x(x^2 + 1)\,dx = \frac{1}{2}x^2\left(\frac{1}{3}x^3 + x\right) + C$

93. $\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10} (bx^2 + d)\,dx$

94. $\int_a^b \sin x\,dx = \int_a^{b+2\pi} \sin x\,dx$

95. $4\int \sin x \cos x\,dx = -\cos 2x + C$

96. $\int \sin^2 2x \cos 2x\,dx = \frac{1}{3}\sin^3 2x + C$

97. Rewriting Integrals Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

98. Integration and Differentiation

(a) Verify that $\sin u - u \cos u + C = \int u \sin u\,du$.

(b) Use part (a) to show that $\int_0^{\pi^2} \sin\sqrt{x}\,dx = 2\pi$.

99. Proof Complete the proof of Theorem 4.16.

100. Rewriting Integrals Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x + h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

PUTNAM EXAM CHALLENGE

101. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n + 1} = 0,$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real root.

102. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

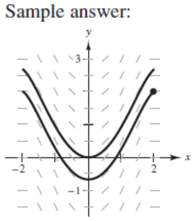
where α is a given real number.

Answers

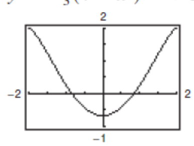
Section 4.5 *(page 301)*

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2 + 1)^2(16x)\,dx$	$8x^2 + 1$	$16x\,dx$
3. $\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

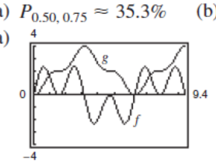
5. $\frac{1}{5}(1 + 6x)^5 + C$ **7.** $\frac{2}{3}(25 - x^2)^{3/2} + C$
9. $\frac{1}{12}(x^4 + 3)^3 + C$ **11.** $\frac{1}{15}(x^3 - 1)^5 + C$
13. $\frac{1}{3}(t^2 + 2)^{3/2} + C$ **15.** $-\frac{15}{8}(1 - x^2)^{4/3} + C$
17. $1/[4(1 - x^2)^2] + C$ **19.** $-1/[3(1 + x^3)] + C$
21. $-\sqrt{1 - x^2} + C$ **23.** $-\frac{1}{4}(1 + 1/t)^4 + C$
25. $\sqrt{2x} + C$ **27.** $2x^2 - 4\sqrt{16 - x^2} + C$
29. $-1/[2(x^2 + 2x - 3)] + C$
31. (a) Answers will vary. Sample answer:



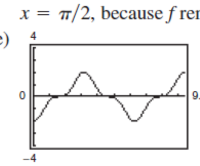
(b) $y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$



33. $-\cos(\pi x) + C$
35. $\int \cos 8x\,dx = \frac{1}{8}\int (\cos 8x)(8)\,dx = \frac{1}{8}\sin 8x + C$
37. $-\sin(1/\theta) + C$
39. $\frac{1}{4}\sin^2 2x + C$ or $-\frac{1}{4}\cos^2 2x + C_1$ or $-\frac{1}{8}\cos 4x + C_2$
41. $\frac{1}{2}\tan^2 x + C$ or $\frac{1}{2}\sec^2 x + C_1$ **43.** $f(x) = 2\cos(x/2) + 4$
45. $f(x) = \frac{1}{12}(4x^2 - 10)^3 - 8$
47. $\frac{2}{5}(x + 6)^{5/2} - 4(x + 6)^{3/2} + C = \frac{2}{5}(x + 6)^{3/2}(x - 4) + C$
49. $-\left[\frac{2}{3}(1 - x)^{3/2} - \frac{4}{5}(1 - x)^{5/2} + \frac{2}{7}(1 - x)^{7/2}\right] + C = -\frac{2}{105}(1 - x)^{3/2}(15x^2 + 12x + 8) + C$
51. $\frac{1}{8}\left[\frac{2}{5}(2x - 1)^{5/2} + \frac{4}{3}(2x - 1)^{3/2} - 6(2x - 1)^{1/2}\right] + C = \left(\sqrt{2x - 1}/15\right)(3x^2 + 2x - 13) + C$
53. $-x - 1 - 2\sqrt{x + 1} + C$ or $-\left(x + 2\sqrt{x + 1}\right) + C_1$
55. 0 **57.** $12 - \frac{8}{9}\sqrt{2}$ **59.** 2 **61.** $\frac{1}{2}$
63. $f(x) = (2x^3 + 1)^3 + 3$ **65.** $1209/28$ **67.** $2\left(\sqrt{3} - 1\right)$
69. $\frac{272}{15}$ **71.** $\frac{2}{3}$ **73.** (a) $\frac{64}{3}$ (b) $\frac{128}{3}$ (c) $-\frac{64}{3}$ (d) 64
75. $2\int_0^3 (4x^2 - 6)\,dx = 36$
77. If $u = 5 - x^2$, the $du = -2x\,dx$ and $\int x(5 - x^2)^3\,dx = -\frac{1}{2}\int(5 - x^2)^3(-2x)\,dx = -\frac{1}{2}\int u^3\,du.$
79. (a) $\int x^2\sqrt{x^3 + 1}\,dx$ (b) $\int \tan(3x)\sec^2(3x)\,dx$
81. \$340,000
83. (a) 102.532 thousand units (b) 102.352 thousand units
(c) 74.5 thousand units
85. (a) $P_{0.50,\,0.75} \approx 35.3\%$ (b) $b \approx 58.6\%$
87. (a)



(b) g is nonnegative, because the graph of f is positive at the beginning and generally has more positive sections than negative ones.
(c) The points on g that correspond to the extrema of f are points of inflection of g.
(d) No, some zeros of f, such as $x = \pi/2$, do not correspond to extrema of g. The graph of g continues to increase after $x = \pi/2$, because f remains above the x-axis.
(e)



The graph of h is that of g shifted 2 units downward.

89. (a) and (b) Proofs
91. False. $\int (2x + 1)^2\,dx = \frac{1}{6}(2x + 1)^3 + C$ **93.** True
95. True **97–99.** Proofs **101.** Putnam Problem A1, 1958