6.2 Integration by Substitution

In problems 1 through 8, find the indicated integral.

1. $\int (2x+6)^5 dx$
   Solution. Substituting $u = 2x + 6$ and $\frac{1}{2}du = dx$, you get

$$\int (2x+6)^5 dx = \frac{1}{2}\int u^5 du = \frac{1}{12}u^6 + C = \frac{1}{12}(2x+6)^6 + C.$$

2. $\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx$
   Solution. Substituting $u = x - 1$ and $du = dx$, you get

$$\begin{aligned}\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx &= \int (u^5 + 3u^2 + 5)du = \\ &= \frac{1}{6}u^6 + u^3 + 5u + C = \\ &= \frac{1}{6}(x-1)^6 + (x-1)^3 + 5(x-1) + C.\end{aligned}$$

   Since, for a constant C, $C - 5$ is again a constant, you can write

$$\int \left[(x-1)^5 + 3(x-1)^2 + 5\right] dx = \frac{1}{6}(x-1)^6 + (x-1)^3 + 5x + C.$$

3. $\int xe^{x^2}dx$
Solution. Substituting $u = x^2$ and $\frac{1}{2}du = xdx$, you get

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int x^5 e^{1-x^6}dx$
Solution. Substituting $u = 1 - x^6$ and $-\frac{1}{6}du = x^5 dx$, you get

$$\int x^5 e^{1-x^6}dx = -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{1-x^6} + C.$$

5. $\int \frac{2x^4}{x^5+1}dx$
Solution. Substituting $u = x^5 + 1$ and $\frac{2}{5}du = 2x^4 dx$, you get

$$\int \frac{2x^4}{x^5+1}dx = \frac{2}{5}\int \frac{1}{u}du = \frac{2}{5}\ln|u| + C = \frac{2}{5}\ln\left|x^5+1\right| + C.$$

6. $\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx$
Solution. Substituting $u = x^4 - x^2 + 6$ and $\frac{5}{2}du = (10x^3 - 5x)dx$, you get

$$\begin{aligned}\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx &= \frac{5}{2}\int \frac{1}{\sqrt{u}}du = \frac{5}{2}\int u^{-\frac{1}{2}}du = \frac{5}{2}\cdot 2u^{\frac{1}{2}} + C = \\ &= 5\sqrt{x^4-x^2+6} + C.\end{aligned}$$

7. $\int \frac{1}{x\ln x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$$

8. $\int \frac{\ln x^2}{x} dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{\ln x^2}{x} dx = \int \frac{2\ln x}{x} dx = 2\int u du = 2 \cdot \frac{1}{2}u^2 + C = (\ln x)^2 + C.$$

9. Use an appropriate change of variables to find the integral

$$\int (x+1)(x-2)^9 dx.$$

Solution. Substituting $u = x - 2$, $u + 3 = x + 1$ and $du = dx$, you get

$$\begin{aligned}\int (x+1)(x-2)^9 dx &= \int (u+3)u^9 du = \int (u^{10} + 3u^9) du = \\ &= \frac{1}{11}u^{11} + \frac{3}{10}u^{10} + C = \\ &= \frac{1}{11}(x-2)^{11} + \frac{3}{10}(x-2)^{10} + C.\end{aligned}$$

10. Use an appropriate change of variables to find the integral

$$\int (2x+3)\sqrt{2x-1}dx.$$

Solution. Substituting $u = 2x - 1$, $u + 4 = 2x + 3$ and $\frac{1}{2}du = dx$, you get

$$\begin{aligned}
\int (2x+3)\sqrt{2x-1}dx &= \frac{1}{2}\int (u+4)\sqrt{u}du = \frac{1}{2}\int u^{\frac{3}{2}}du + 2\int u^{\frac{1}{2}}du = \\
&= \frac{1}{2}\cdot\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^{\frac{5}{2}} + \frac{4}{3}(2x-1)^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^2\sqrt{2x-1} + \frac{4}{3}(2x-1)\sqrt{2x-1} + C = \\
&= (2x-1)\sqrt{2x-1}\left(\frac{2}{5}x - \frac{1}{5} + \frac{4}{3}\right) + C = \\
&= \left(\frac{2}{5}x + \frac{17}{25}\right)(2x-1)\sqrt{2x-1} + C.
\end{aligned}$$

Homework

In problems 1 through 18, find the indicated integral and check your answer by differentiation.

1. $\int e^{5x} dx$
2. $\int \sqrt{4x-1} dx$
3. $\int \frac{1}{3x+5} dx$
4. $\int e^{1-x} dx$
5. $\int 2xe^{x^2-1} dx$
6. $\int x(x^2+1)^5 dx$
7. $\int 3x\sqrt{x^2+8} dx$
8. $\int x^2(x^3+1)^{\frac{3}{4}} dx$
9. $\int \frac{x^2}{(x^3+5)^2} dx$
10. $\int (x+1)(x^2+2x+5)^{12} dx$
11. $\int (3x^2-1)e^{x^3-x} dx$
12. $\int \frac{3x^4+12x^3+6}{x^5+5x^4+10x+12} dx$
13. $\int \frac{3x-3}{(x^2-2x+6)^2} dx$
14. $\int \frac{6x-3}{4x^2-4x+1} dx$
15. $\int \frac{\ln 5x}{x} dx$
16. $\int \frac{1}{x(\ln x)^2} dx$
17. $\int \frac{2x\ln(x^2+1)}{x^2+1} dx$
18. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

In problems 19 through 23, use an appropriate change of variables to find the indicated integral.

19. $\int \frac{x}{x-1} dx$
20. $\int x\sqrt{x+1} dx$
21. $\int \frac{x}{(x-5)^6} dx$
22. $\int \frac{x+3}{(x-4)^2} dx$
23. $\int \frac{x}{2x+1} dx$

24. Find the function whose tangent has slope $x\sqrt{x^2+5}$ for each value of x and whose graph passes through the point $(2, 10)$.

25. Find the function whose tangent has slope $\frac{2x}{1-3x^2}$ for each value of x and whose graph passes through the point $(0, 5)$.

In problems 19 through 23, use an appropriate change of variables to find the indicated integral.

19. $\int \frac{x}{x-1} dx$ 20. $\int x\sqrt{x+1} dx$

21. $\int \frac{x}{(x-5)^6} dx$ 22. $\int \frac{x+3}{(x-4)^2} dx$

23. $\int \frac{x}{2x+1} dx$

24. Find the function whose tangent has slope $x\sqrt{x^2+5}$ for each value of x and whose graph passes through the point $(2, 10)$.

25. Find the function whose tangent has slope $\frac{2x}{1-3x^2}$ for each value of x and whose graph passes through the point $(0, 5)$.

26. A tree has been transplanted and after x years is growing at the rate of $1 + \frac{1}{(x+1)^2}$ meters per year. After two years it has reached a height of five meters. How tall was it when it was transplanted?

27. It is projected that t years from now the population of a certain country will be changing at the rate of $e^{0.02t}$ million per year. If the current population is 50 million, what will the population be 10 years from now?

Answers

1. $\frac{1}{5}e^{5x}+C$
2. $\frac{1}{6}(4x-1)\sqrt{4x-1}+C$
3. $\frac{1}{3}\ln|3x+5|+C$
4. $-e^{1-x}+C$
5. $e^{x^2-1}+C$
6. $\frac{1}{12}(x^2+1)^6+C$
7. $(x^2+8)\sqrt{x^2+8}+C$
8. $\frac{4}{21}(x^3+1)^{\frac{7}{4}}+C$
9. $-\frac{1}{3(x^3+5)}+C$
10. $\frac{1}{26}(x^2+2x+5)^{13}+C$
11. $e^{x^3-x}+C$
12. $\frac{3}{5}\ln|x^5+5x^4+10x+12|+C$
13. $-\frac{3}{2(x^2-2x+6)}+C$
14. $\frac{3}{2}\ln|2x-1|+C$
15. $\frac{1}{2}\ln^2 5x+C$
16. $-\frac{1}{\ln x}+C$
17. $\frac{1}{2}\ln^2(x^2+1)+C$
18. $2e^{\sqrt{x}}+C$
19. $x+\ln|x-1|+C$
20. $\frac{2}{5}(x+1)^2\sqrt{x+1}-\frac{2}{3}(x+1)\sqrt{x+1}+C$
21. $-\frac{1}{(x-5)^5}-\frac{1}{4(x-5)^4}+C$
22. $-\frac{7}{x-4}+\ln|x-4|+C$
23. $\frac{2x+1}{4}x-\frac{1}{4}\ln|2x+1|+C$

24. $f(x)=\frac{1}{3}(x^2+5)\sqrt{x^2+5}+1$

25. $f(x)=-\frac{1}{3}\ln|1-3x^2|+5$

26. $\frac{7}{3}$ meters

27. 61 million