1 Integration By Substitution (Change of Variables)

We can think of integration by substitution as the counterpart of the chain rule for differentiation. Suppose that $g(x)$ is a differentiable function and f is continuous on the range of g. Integration by substitution is given by the following formulas:

Indefinite Integral Version:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{where } u = g(x).$$

Definite Integral Version:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{where } u = g(x).$$

1.1 Example Problems

Strategy: The idea is to make the integral easier to compute by doing a change of variables.

1. Start by guessing what the appropriate change of variable $u = g(x)$ should be. Usually you choose u to be the function that is "inside" the function.

2. Differentiate both sides of $u = g(x)$ to conclude $du = g'(x)dx$. If we have a definite integral, use the fact that $x = a \to u = g(a)$ and $x = b \to u = g(b)$ to also change the bounds of integration.

3. Rewrite the integral by replacing all instances of x with the new variable and compute the integral or definite integral.

4. If you computed the indefinite integral, then make sure to write your final answer back in terms of the original variables.

Problem 1. ($\star$) Find
$$\int \tan(x)\, dx.$$

Solution 1.

Step 1: We will use the change of variables $u = \cos(x)$,
$$\frac{du}{dx} = -\sin(x) \Rightarrow du = -\sin(x)\, dx.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Problem 2. ($\star$) Find
$$\int_0^1 xe^{-\frac{x^2}{2}}\, dx.$$

Solution 2.

Step 1: We will use the change of variables $u = -\frac{x^2}{2}$,
$$\frac{du}{dx} = -x \Rightarrow du = -x\, dx, \quad x = 0 \to u = 0, \quad x = 1 \to u = -\frac{1}{2}.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\int xe^{\frac{-x^2}{2}}\, dx = -\int_0^{-\frac{1}{2}} e^u\, du = -e^u\Big|_{u=0}^{u=-\frac{1}{2}} = -e^{-\frac{1}{2}} + 1.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,
$$\int xe^{-\frac{x^2}{2}}\, dx = -e^{-\frac{x^2}{2}},$$
then use the fundamental theorem of calculus to conclude
$$\int_0^1 xe^{-\frac{x^2}{2}}\, dx = -e^{-\frac{x^2}{2}}\Big|_{x=0}^{x=1} = -e^{-\frac{1}{2}} + 1.$$

Problem 3. ($\star$) Find

$$\int \tanh(x)\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$$

Solution 3.

Step 1: We will use the change of variables $u = e^x + e^{-x}$,

$$\frac{du}{dx} = e^x - e^{-x} \Rightarrow du = (e^x - e^{-x})\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{du}{u} &= \ln|u| + C \\ &= \ln|e^x + e^{-x}| + C. \qquad u = e^x + e^{-x}\end{aligned}$$

Since $e^x + e^{-x} > 0$, we can remove the absolute values if we wish giving the final answer

$$\int \tanh(x)\,dx = \ln(e^x + e^{-x}) + C.$$

Remark: We can use the fact $e^x + e^{-x} = 2\cosh(x)$ to conclude that

$$\ln(e^x + e^{-x}) + C = \ln(2\cosh(x)) + C = \ln(\cosh(x)) + \underbrace{\ln(2) + C}_{D} = \ln(\cosh(x)) + D.$$

This form of the indefinite integral may be easier to remember since it mirrors the fact that

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C.$$

Problem 4. ($\star$) Evaluate

$$\int_0^1 x\sqrt{1-x^2}\,dx.$$

Solution 4.

Step 1: We will use the change of variables $u = 1 - x^2$,

$$\frac{du}{dx} = -2x \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = xdx, \qquad x = 0 \to u = 1, \qquad x = 1 \to u = 0.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=0} = \frac{1}{3}.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}},$$

then use the fundamental theorem of calculus to conclude

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Problem 5. ($\star\star$) Find

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

Solution 5.

Step 1: We will use the change of variables $u = \sqrt{x}$,

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}du = dx \Rightarrow 2u\,du = dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du.$$

This integral is a bit tricky to compute, so we have to use algebra to simplify it first. Using long division to first simplify the integrand, we get

$$\begin{aligned}\int \frac{2u}{1+u}\,du = 2\int \frac{u}{1+u}\,du = 2\int 1 - \frac{1}{1+u}\,du \\ = 2u - 2\ln|1+u| + C \\ = 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C. \qquad u = \sqrt{x}.\end{aligned}$$

Alternative Solution: We can also do a change of variables by writing x as a function of u.

Step 1: We can also do the change of variables $x = u^2$,

$$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+\sqrt{u^2}}\,du = \int \frac{2u}{1+u}\,du.$$

The computation is now identical to the case above.

Problem 6. (⋆⋆) Find
$$\int \sec(x)\,dx.$$

Solution 6. We first do a trick by multiplying the numerator and denominator by $\sec(x) + \tan(x)$,
$$\int \sec(x)\,dx = \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Step 1: We will use the change of variables $u = \sec(x) + \tan(x)$,
$$\frac{du}{dx} = \sec(x)\tan(x) + \sec^2(x) \Rightarrow du = (\sec(x)\tan(x) + \sec^2(x))\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\begin{aligned}\int \sec(x)\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x))}{\sec(x)+\tan(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\sec(x)+\tan(x)| + C. \qquad u = \sec(x)+\tan(x)\end{aligned}$$

Problem 7. (⋆⋆) Find
$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx.$$

Solution 7.

Step 1: We will use the change of variables $u = e^x$,
$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}\,du \Rightarrow dx = \frac{1}{u}\,du.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx &= \int \frac{2}{u(u+u^{-1})}\,du \\ &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

Alternative Solution: We first do a trick by multiplying the numerator and denominator by e^x,
$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx.$$

Step 1: We will use the change of variables $u = e^x$,

$$\frac{du}{dx} = e^x \Rightarrow du = e^x\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

1.1.1 Proofs of the Symmetry Properties of Integration

Problem 1. ($\star\star\star$) Suppose that $f(-x) = f(x)$. Prove that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Solution 1. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for even function f,

$$\begin{aligned}\int_0^{-a} f(x)\,dx &= -\int_0^a f(-u)\,du \qquad u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a \\ &= -\int_0^a f(u)\,du \qquad f(-x)=f(x) \\ &= -\int_0^a f(x)\,dx.\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Problem 3. ($\star\star$) Suppose $f(x)$ is even ($f(-x) = f(x)$). Show that the function

$$F(x) = \int_0^x f(t)\,dt$$

is an odd function.

Solution 3. It suffices to show $F(-x) = -F(x)$. Using the change of variables $u = -t$,

$$du = -dt, \qquad t = 0 \to u = 0, \qquad t = -x \to u = x$$

we have

$$\begin{aligned} F(-x) = \int_0^{-x} f(t)\,dt &= -\int_0^x f(-u)\,du \\ &= -\int_0^x f(u)\,du \qquad f(-u) = f(u) \\ &= -F(x). \end{aligned}$$

Problem 4. ($\star\star$) Suppose $f(x)$ is odd ($f(-x) = -f(x)$). Show that the function

$$F(x) = \int_a^x f(t)\,dt$$

is an even function.

Solution 4. It suffices to show $F(-x) = F(x)$. Using the change of variables $u = -t$,

$$du = -dt, \qquad t = a \to u = -a, \qquad t = -x \to u = x$$

we have

$$F(-x) = \int_a^{-x} f(t)\,dt = -\int_{-a}^x f(-u)\,du = \int_{-a}^x f(u)\,du. \qquad f(-u) = -f(u)$$

It may appear that the last term is not of the same form as the term $F(x)$ because the lower bounds of integration are different. However, we can split the region of integration and use a change of variables to conclude that

$$\begin{aligned} \int_{-a}^x f(u)\,du &= \int_{-a}^0 f(u)\,du + \int_0^x f(u)\,du \\ &= -\int_a^0 f(-\tilde{u})\,d\tilde{u} + \int_0^x f(u)\,du \qquad \tilde{u} = -u, d\tilde{u} = -du, \int_{-a}^0 du \to \int_a^0 d\tilde{u} \\ &= \int_a^0 f(\tilde{u})\,d\tilde{u} + \int_0^x f(u)\,du \qquad f(-u) = -f(u) \\ &= \int_a^x f(t)\,dt = F(x). \end{aligned}$$

Remark: If we use the result from Problem 2 on Page 5, then we have the shorter proof,

$$F(-x) = \int_a^{-x} f(t)\,dt = -\int_{-a}^x f(-u)\,du = \int_{-a}^x f(u)\,du = \underbrace{\int_{-a}^a f(u)\,du}_{=0} + \int_a^x f(u)\,du = F(x).$$