7.2 Integration by Substitution

Integrating the chain rule leads to the method of substitution.

The method of integration by substitution is based on the chain rule for differentiation. If F and g are differentiable functions, the chain rule tells us that $(F \circ g)'(x) = F'(g(x))g'(x)$; that is, $F(g(x))$ is an antiderivative of $F'(g(x))g'(x)$. In indefinite integral notation, we have

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

As in differentiation, it is convenient to introduce an intermediate variable $u = g(x)$; then the preceding formula becomes

$$\int F'(u)\frac{du}{dx}\,dx = F(u) + C.$$

If we write $f(u)$ for $F'(u)$, so that $\int f(u)\,du = F(u) + C$, we obtain the formula

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du. \qquad (1)$$

This formula is easy to remember, since one may "cancel the dx's."

To apply the method of substitution one must find in a given integrand an expression $u = g(x)$ whose derivative $du/dx = g'(x)$ also occurs in the integrand.

Example 1 Find $\int 2x\sqrt{x^2+1}\,dx$ and check the answer by differentiation.

Solution None of the rules in Section 7.1 apply to this integral, so we try integration by substitution. Noticing that $2x$, the derivative of x^2+1, occurs in the integrand, we are led to write $u = x^2 + 1$; then we have

$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\cdot 2x\,dx = \int \sqrt{u}\left(\frac{du}{dx}\right)dx.$$

By formula (1), the last integral equals $\int \sqrt{u}\,du = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C$. At this point we substitute x^2+1 for u, which gives

$$\int 2x\sqrt{x^2+1}\,dx = \tfrac{2}{3}(x^2+1)^{3/2} + C.$$

Checking our answer by differentiating has educational as well as insurance value, since it will show how the chain rule produces the integrand we started with:

$$\frac{d}{dx}\left[\frac{2}{3}(x^2+1)^{3/2} + C\right] = \frac{2}{3}\cdot\frac{3}{2}(x^2+1)^{1/2}\frac{d}{dx}(x^2+1) = \left[\sqrt{x^2+1}\,\right]2x,$$

as it should be. ▲

Sometimes the derivative of the intermediate variable is "hidden" in the integrand. If we are clever, however, we can still use the method of substitution, as the next example shows.

Example 2 Find $\int \cos^2 x \sin x\,dx$.

Solution We are tempted to make the substitution $u = \cos x$, but du/dx is then $-\sin x$ rather than $\sin x$. No matter—we can rewrite the integral as

$$\int (-\cos^2 x)(-\sin x)\,dx.$$

Setting $u = \cos x$, we have

$$\int -u^2\frac{du}{dx}\,dx = \int -u^2\,du = -\frac{u^3}{3} + C,$$

so

$$\int \cos^2 x \sin x\,dx = -\tfrac{1}{3}\cos^3 x + C.$$

You may check this by differentiating. ▲

Example 3 Find $\int \frac{e^x}{1 + e^{2x}}\, dx$.

Solution We cannot just let $u = 1 + e^{2x}$, because $du/dx = 2e^{2x} \neq e^x$; but we may recognize that $e^{2x} = (e^x)^2$ and remember that the derivative of e^x is e^x. Making the substitution $u = e^x$ and $du/dx = e^x$, we have

$$\int \frac{e^x}{1 + e^{2x}}\, dx = \int \frac{1}{1 + (e^x)^2} \cdot e^x\, dx$$

$$= \int \frac{1}{1 + u^2} \cdot \frac{du}{dx} \cdot dx = \int \frac{1}{1 + u^2}\, du$$

$$= \tan^{-1} u + C = \tan^{-1}(e^x) + C.$$

Again you should check this by differentiation. ▲

We may summarize the method of substitution as developed so far (see Fig. 7.2.1).

Integration by Substitution

To integrate a function which involves an intermediate variable u and its derivative du/dx, write the integrand in the form $f(u)(du/dx)$, incorporating constant factors as required in $f(u)$. Then apply the formula

$$\int f(u)\, \frac{du}{dx}\, dx = \int f(u)\, du.$$

Finally, evaluate $\int f(u)\, du$ if you can; then substitute for u its expression in terms of x.

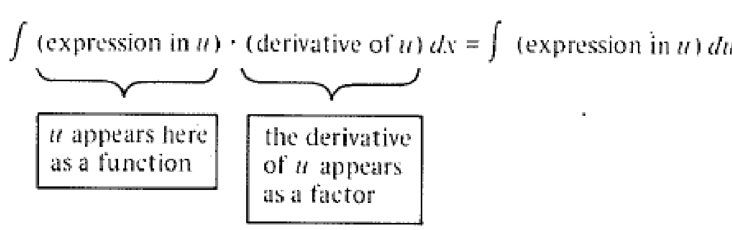

Figure 7.2.1. How to spot u in a substitution problem.

Example 4 Find (a) $\int x^2 \sin(x^3)\,dx$, (b) $\int \sin 2x\,dx$.

Solution (a) We observe that the factor x^2 is, apart from a factor of 3, the derivative of x^3. Substitute $u = x^3$, so $du/dx = 3x^2$ and $x^2 = \frac{1}{3}\,du/dx$. Thus

$$\int x^2 \sin(x^3)\,dx = \int \frac{1}{3}\frac{du}{dx}\sin u\,dx = \frac{1}{3}\int (\sin u)\frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C.$$

Hence $\int x^2 \sin(x^3)\,dx = -\frac{1}{3}\cos(x^3) + C$.
(b) Substitute $u = 2x$, so $du/dx = 2$. Then

$$\int \sin 2x\,dx = \int \frac{1}{2}(\sin 2x)2\,dx = \frac{1}{2}\int \sin u\frac{du}{dx}\,dx$$
$$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C.$$

Thus

$$\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C. \blacktriangle$$

Example 5 Evaluate: (a) $\int \frac{x^2}{x^3+5}\,dx$, (b) $\int \frac{dt}{t^2-6t+10}$ [*Hint*: Complete the square in the denominator], and (c) $\int \sin^2 2x \cos 2x\,dx$.

Solution (a) Set $u = x^3 + 5$; $du/dx = 3x^2$. Then

$$\int \frac{x^2}{x^3+5}\,dx = \int \frac{1}{3(x^3+5)}\,3x^2\,dx = \frac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+5| + C.$$

(b) Completing the square (see Section R.1), we find

$$t^2 - 6t + 10 = (t^2 - 6t + 9) - 9 + 10$$

$$= (t-3)^2 + 1$$

We set $u = t - 3$; $du/dt = 1$. Then

$$\int \frac{dt}{t^2-6t+10} = \int \frac{dt}{1+(t-3)^2} = \int \frac{1}{1+u^2}\frac{du}{dt}\,dt$$

$$= \int \frac{1}{1+u^2}\,du = \tan^{-1}u + C,$$

so

$$\int \frac{dt}{t^2-6t+10} = \tan^{-1}(t-3) + C.$$

(c) We first substitute $u = 2x$, as in Example 4(b). Since $du/dx = 2$,

$$\int \sin^2 2x \cos 2x\,dx = \int \sin^2 u \cos u \frac{1}{2}\frac{du}{dx}\,dx = \frac{1}{2}\int \sin^2 u \cos u\,du.$$

At this point, we notice that another substitution is appropriate: we set $s = \sin u$ and $ds/du = \cos u$. Then

$$\frac{1}{2}\int \sin^2 u \cos u \, du = \frac{1}{2}\int s^2 \frac{ds}{du}\, du = \frac{1}{2}\int s^2\, ds$$

$$= \frac{1}{2}\cdot\frac{1}{3}s^3 + C = \frac{s^3}{6} + C.$$

Now we must put our answer in terms of x. Since $s = \sin u$ and $u = 2x$, we have

$$\int \sin^2 2x \cos 2x \, dx = \frac{s^3}{6} + C = \frac{\sin^3 u}{6} + C = \frac{\sin^3 2x}{6} + C.$$

You should check this formula by differentiating.

You may have noticed that we could have done this problem in one step by substituting $u = \sin 2x$ in the beginning. We did the problem the long way to show that you can solve an integration problem even if you do not see everything at once. ▲

Two simple substitutions are so useful that they are worth noting explicitly. We have already used them in the preceding examples. The first is the *shifting rule*, obtained by the substitution $u = x + a$, where a is a constant. Here $du/dx = 1$.

Shifting Rule

To evaluate $\int f(x+a)\,dx$, first evaluate $\int f(u)\,du$, then substitute $x + a$ for u:

$$\int f(x+a)\,dx = F(x+a) + C, \qquad \text{where } F(u) = \int f(u)\,du.$$

The second rule is the *scaling rule*, obtained by substituting $u = bx$, where b is a constant. Here $du/dx = b$. The substitution corresponds to a change of scale on the x axis.

Scaling Rule

To evaluate $\int f(bx)\,dx$, evaluate $\int f(u)\,du$, divide by b and substitute bx for u:

$$\int f(bx)\,dx = \frac{1}{b}F(bx) + C, \qquad \text{where } F(u) = \int f(u)\,du.$$

Example 6 Find (a) $\int \sec^2(x+7)\,dx$ and (b) $\int \cos 10x\,dx$.

Solution (a) Since $\int \sec^2 u\,du = \tan u + C$, the shifting rule gives

$$\int \sec^2(x+7)\,dx = \tan(x+7) + C.$$

(b) Since $\int \cos u\,du = \sin u + C$, the scaling rule gives

$$\int \cos 10x\,dx = \tfrac{1}{10}\sin(10x) + C. \blacktriangle$$

You do not need to memorize the shifting and scaling rules as such; however, the underlying substitutions are so common that you should learn to use them rapidly and accurately.

To conclude this section, we shall introduce a useful device called *differential notation*, which makes the substitution process more mechanical. In particular, this notation helps keep track of the constant factors which must be distributed between the $f(u)$ and du/dx parts of the integrand. We illustrate the device with an example before explaining why it works.

Example 7 Find $\int \frac{x^4+2}{(x^5+10x)^5}\,dx$.

Solution We wish to substitute $u = x^5 + 10x$; note that $du/dx = 5x^4 + 10$. Pretending that du/dx is a fraction, we may "solve for dx," writing $dx = du/(5x^4 + 10)$. Now we substitute u for $x^5 + 10x$ and $du/(5x^4 + 10)$ for dx in our integral to obtain

$$\int \frac{x^4+2}{(x^5+10x)^5}\,dx = \int \frac{x^4+2}{u^5}\,\frac{du}{5x^4+10} = \int \frac{x^4+2}{5(x^4+2)}\,\frac{du}{u^5} = \int \frac{1}{5}\,\frac{du}{u^5}\,.$$

Notice that the $(x^4 + 2)$'s cancelled, leaving us an integral in u which we can evaluate:

$$\frac{1}{5}\int \frac{du}{u^5} = \frac{1}{5}\left(-\frac{1}{4}u^{-4}\right) + C = -\frac{1}{20u^4} + C.$$

Substituting $x^5 + 10x$ for u gives

$$\int \frac{x^4+2}{(x^5+10x)^5}\,dx = -\frac{1}{20(x^5+10x)^4} + C. \blacktriangle$$

Although du/dx is not really a fraction, we can still justify "solving for dx" when we integrate by substitution. Suppose that we are trying to integrate $\int h(x)\,dx$ by substituting $u = g(x)$. Solving $du/dx = g'(x)$ for dx amounts to replacing dx by $du/g'(x)$ and hence writing

$$\int h(x)\,dx = \int \frac{h(x)}{g'(x)}\,du. \tag{2}$$

Now suppose that we can express $h(x)/g'(x)$ in terms of u, i.e., $h(x)/g'(x) = f(u)$ for some function f. Then we are saying that $h(x) = f(u)g'(x) = f(g(x))g'(x)$, and equation (2) just says

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$

which is the form of integration by substitution we have been using all along.

Example 8 Find $\int \left(\frac{e^{1/x}}{x^2} \right) dx.$

Solution Let $u = 1/x$; $du/dx = -1/x^2$ and $dx = -x^2\,du$, so

$$\int \left(\frac{1}{x^2} \right) e^{1/x}\,dx = \int \left(\frac{1}{x^2} \right) e^{u}(-x^2\,du) = -\int e^u\,du = -e^u + C$$

and therefore

$$\int \left(\frac{1}{x^2} \right) e^{1/x}\,dx = -e^{1/x} + C. \blacktriangle$$

Integration by Substitution (Differential Notation)

To integrate $\int h(x)\,dx$ by substitution:

1. Choose a new variable $u = g(x)$.
2. Differentiate to get $du/dx = g'(x)$ and then solve for dx.
3. Replace dx in the integral by the expression found in step 2.
4. Try to express the new integrand completely in terms of u, eliminating x. (If you cannot, try another substitution or another method.)
5. Evaluate the new integral $\int f(u)\,du$ (if you can).
6. Express the result in terms of x.
7. Check by differentiating.

Example 9 (a) Calculate the following integrals: (a) $\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx$,

(b) $\int \cos x[\cos(\sin x)]\,dx$, and (c) $\int \left(\frac{\sqrt{1+\ln x}}{x}\right)dx$.

Solution (a) Let $u = x^3 + 3x^2 + 1$; $du/dx = 3x^2 + 6x$, so $dx = du/(3x^2 + 6x)$ and

$$\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx = \int \frac{1}{\sqrt[3]{u}}\,\frac{x^2+2x}{3x^2+6x}\,du$$

$$= \frac{1}{3}\int \frac{1}{\sqrt[3]{u}}\,du = \frac{1}{3}\cdot\frac{3}{2}u^{2/3} + C.$$

Thus

$$\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}\,dx = \frac{1}{2}(x^3+3x^2+1)^{2/3} + C.$$

(b) Let $u = \sin x$; $du/dx = \cos x$, $dx = du/\cos x$, so

$$\int \cos x\big[\cos(\sin x)\big]\,dx = \int \cos x\big[\cos(\sin x)\big]\frac{du}{\cos x}$$

$$= \int \cos u\,du = \sin u + C,$$

and therefore

$$\int \cos x\big[\cos(\sin x)\big]\,dx = \sin(\sin x) + C.$$

(c) Let $u = 1 + \ln x$; $du/dx = 1/x$, $dx = x\,du$, so

$$\int \frac{\sqrt{1+\ln x}}{x}\,dx = \int \frac{\sqrt{1+\ln x}}{x}(x\,du) = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C,$$

and therefore

$$\int \frac{\sqrt{1+\ln x}}{x}\,dx = \frac{2}{3}(1+\ln x)^{3/2} + C. \blacktriangle$$

Exercises for Section 7.2

Evaluate each of the integrals in Exercises 1–6 by making the indicated substitution, and check your answers by differentiating.

1. $\int 2x(x^2+4)^{3/2}\,dx;\ u = x^2+4.$
2. $\int (x+1)(x^2+2x-4)^{-4}\,dx;\ u = x^2+2x-4.$
3. $\int \frac{2y^7+1}{(y^8+4y-1)^2}\,dy;\ x = y^8+4y-1.$
4. $\int \frac{x}{1+x^4}\,dx;\ u = x^2.$
5. $\int \frac{\sec^2\theta}{\tan^3\theta}\,d\theta;\ u = \tan\theta.$
6. $\int \tan x\,dx;\ u = \cos x.$

Evaluate each of the integrals in Exercises 7–22 by the method of substitution, and check your answer by differentiating.

7. $\int (x+1)\cos(x^2+2x)\,dx$
8. $\int u\sin(u^2)\,du$
9. $\int \frac{x^3}{\sqrt{x^4+2}}\,dx$
10. $\int \frac{x}{(x^2+3)^2}\,dx$
11. $\int \frac{t^{1/3}}{(t^{4/3}+1)^{3/2}}\,dt$
12. $\int \frac{x^{1/2}}{(x^{3/2}+2)^2}\,dx$
13. $\int 2r\sin(r^2)\cos^3(r^2)\,dr$
14. $\int e^{\sin x}\cos x\,dx$
15. $\int \frac{x^3}{1+x^8}\,dx$
16. $\int \frac{dx}{\sqrt{1-4x^2}}$
17. $\int \sin(\theta+4)\,d\theta$
18. $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$
19. $\int (5x^4+1)(x^5+x)^{100}\,dx$
20. $\int (1+\cos s)\sqrt{s+\sin s}\,ds$
21. $\int \left(\frac{t+1}{\sqrt{t^2+2t+3}}\right)dt$
22. $\int \frac{dx}{x^2+4}$

Evaluate the indefinite integrals in Exercises 23–36.

23. $\int t\sqrt{t^2+1}\,dt.$
24. $\int t\sqrt{t+1}\,dt.$
25. $\int \cos^3\theta\,d\theta.$ [*Hint*: Use $\cos^2\theta+\sin^2\theta=1$.]
26. $\int \cot x\,dx.$
27. $\int \frac{dx}{x\ln x}.$
28. $\int \frac{dx}{\ln(x^x)}.$
29. $\int \sqrt{4-x^2}\,dx.$ [*Hint*: Let $x = 2\sin u$.]
30. $\int \sin^2 x\,dx.$ (Use $\cos 2x = 1-2\sin^2 x$.)
31. $\int \frac{\cos\theta}{1+\sin\theta}\,d\theta.$
32. $\int \sec^2 x(e^{\tan x}+1)\,dx.$
33. $\int \frac{\sin(\ln t)}{t}\,dt.$
34. $\int \frac{e^{2s}}{1+e^{2s}}\,ds.$
35. $\int \frac{\sqrt[3]{3+1/x}}{x^2}\,dx.$
36. $\int \frac{1}{x^3}\left(1-\frac{1}{x^2}\right)^{1/3}dx.$
37. Compute $\int \sin x\cos x\,dx$ by each of the following three methods: (a) Substitute $u=\sin x$, (b) substitute $u=\cos x$, (c) use the identity $\sin 2x = 2\sin x\cos x$. Show that the three answers you get are really the same.
38. Compute $\int e^{ax}\,dx$, where a is constant, by each of the following substitutions: (a) $u = ax$; (b) $u = e^x$. Show that you get the same answer either way.

★39. For which values of m and n can $\int \sin^m x\cos^n x\,dx$ be evaluated by using a substitution $u=\sin x$ or $u=\cos x$ and the identity $\cos^2 x+\sin^2 x = 1$?

★40. For which values of r can $\int \tan^r x\,dx$ be evaluated by the substitution suggested in Exercise 39?

7.3 Changing Variables in the Definite Integral

When you change variables in a definite integral, you must keep track of the endpoints.

We have just learned how to evaluate many indefinite integrals by the method of substitution. Using the fundamental theorem of calculus, we can use this knowledge to evaluate definite integrals as well.

Example 1 Find $\int_0^2 \sqrt{x+3}\, dx$.

Solution Substitute $u = x + 3$, $du = dx$. Then

$$\int \sqrt{x+3}\, dx = \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(x+3)^{3/2} + C.$$

By the fundamental theorem of calculus,

$$\int_0^2 \sqrt{x+3}\, dx = \frac{2}{3}(x+3)^{3/2}\Big|_0^2 = \frac{2}{3}\left(5^{3/2} - 3^{3/2}\right) \approx 3.99.$$

To check this result we observe that, on the interval $[0,2]$, $\sqrt{x+3}$ lies between $\sqrt{3}$ (≈ 1.73) and $\sqrt{5}$ (≈ 2.24), so the integral must lie between $2\sqrt{3}$ (≈ 3.46) and $2\sqrt{5}$ (≈ 4.47). (This check actually enabled the authors to spot an error in their first attempted solution of this problem.) ▲

Notice that we must express the indefinite integral in terms of x before plugging in the endpoints 0 and 2, since they refer to values of x. It is possible, however, to evaluate the definite integral directly in the u variable—*provided that we change the endpoints*. We offer an example before stating the general procedure.

Example 2 Find $\int_1^4 \frac{x}{1+x^4}\, dx$.

Solution Substitute $u = x^2$, $du = 2x\, dx$, that is, $x\, dx = du/2$. As x runs from 1 to 4, $u = x^2$ runs from 1 to 16, so we have

$$\int_1^4 \frac{x}{1+x^4}\, dx = \int_1^{16} \frac{x}{1+x^4} \frac{du}{2x} = \frac{1}{2}\int_1^{16} \frac{du}{1+u^2}$$

$$= \frac{1}{2}\tan^{-1}u\Big|_1^{16} = \frac{1}{2}\left(\tan^{-1}16 - \tan^{-1}1\right) \approx 0.361. \ ▲$$

In general, suppose that we have an integral of the form $\int_a^b f(g(x))g'(x)\,dx$. If $F'(u)=f(u)$, then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$; by the fundamental theorem of calculus, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

However, the right-hand side is equal to $\int_{g(a)}^{g(b)} f(u)\,du$, so we have the formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Notice that $g(a)$ and $g(b)$ are the values of $u=g(x)$ when $x=a$ and b, respectively. Thus we can evaluate an integral $\int_a^b h(x)\,dx$ by writing $h(x)$ as

$f(g(x))g'(x)$ and using the formula

$$\int_a^b h(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 3 Evaluate $\int_0^{\pi/4} \cos 2\theta\,d\theta$.

Solution Let $u=2\theta$; $d\theta=\frac{1}{2}\,du$; $u=0$ when $\theta=0$, $u=\pi/2$ when $\theta=\pi/4$. Thus

$$\int_0^{\pi/4} \cos 2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} \cos u\,du = \frac{1}{2}\sin u\Big|_0^{\pi/2} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}\,. \ \blacktriangle$$

Definite Integral by Substitution

Given an integral $\int_a^b h(x)\,dx$ and a new variable $u = g(x)$:

1. Substitute $du/g'(x)$ for dx and then try to express the integrand $h(x)/g'(x)$ in terms of u.
2. Change the endpoints a and b to $g(a)$ and $g(b)$, the corresponding values of u.

Then

$$\int_a^b h(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

where $f(u) = h(x)/(du/dx)$. Since $h(x) = f(g(x))g'(x)$, this can be written as

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 4 Evaluate $\int_1^5 \frac{x}{x^4 + 10x^2 + 25}\,dx$.

Solution Seeing that the denominator can be written in terms of x^2, we try $u = x^2$, $dx = du/(2x)$; $u = 1$ when $x = 1$ and $u = 25$ when $x = 5$. Thus

$$\int_1^5 \frac{x}{x^4 + 10x^2 + 25}\,dx = \frac{1}{2}\int_1^{25} \frac{du}{u^2 + 10u + 25}.$$

Now we notice that the denominator is $(u + 5)^2$, so we set $v = u + 5$, $du = dv$; $v = 6$ when $u = 1$, $v = 30$ when $u = 25$. Therefore

$$\frac{1}{2}\int_1^{25} \frac{du}{u^2 + 10u + 25} = \frac{1}{2}\int_6^{30} \frac{dv}{v^2} = \frac{1}{2}\left(-\frac{1}{v}\right)\Bigg|_6^{30}$$

$$= -\frac{1}{60} + \frac{1}{12} = \frac{1}{15}.$$

If you see the substitution $v = x^2 + 5$ right away, you can do the problem in one step instead of two. ▲

Example 5 Find $\int_0^{\pi/4}(\cos^2\theta - \sin^2\theta)\,d\theta$.

Solution It is not obvious what substitution is appropriate here, so a little trial and error is called for. If we remember the trigonometric identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can proceed easily:

$$\int_0^{\pi/4}(\cos^2\theta - \sin^2\theta)\,d\theta = \int_0^{\pi/4}\cos 2\theta\,d\theta = \int_0^{\pi/2}\cos u\,\frac{du}{2} \qquad (u = 2\theta)$$

$$= \frac{\sin u}{2}\bigg|_0^{\pi/2} = \frac{1-0}{2} = \frac{1}{2}.$$

(See Exercise 32 for another method.) ▲

Example 6 Evaluate $\int_0^1 \frac{e^x}{1+e^x}\,dx$.

Solution Let $u = 1 + e^x$; $du = e^x\,dx$, $dx = du/e^x$; $u = 1 + e^0 = 2$ when $x = 0$ and $u = 1 + e$ when $x = 1$. Thus

$$\int_0^1 \frac{e^x}{1+e^x}\,dx = \int_2^{1+e}\frac{1}{u}\,du = \ln u\bigg|_2^{1+e} = \ln(1+e) - \ln 2 = \ln\left(\frac{1+e}{2}\right).$$ ▲

Substitution does not always work. We can always make a substitution, but sometimes it leads nowhere.

Example 7 What does the integral $\int_0^2 \frac{dx}{1+x^4}$ become if you substitute $u = x^2$?

Solution If $u = x^2$, $du/dx = 2x$ and $dx = du/2x$, so

$$\int_0^2 \frac{dx}{1+x^4} = \int_0^4 \frac{1}{1+u^2}\,\frac{du}{2x}.$$

We must solve $u = x^2$ for x; since $x \geqslant 0$, we get $x = \sqrt{u}$, so

$$\int_0^2 \frac{dx}{1+x^4} = \int_0^4 \frac{du}{2\sqrt{u}\,(1+u^2)}.$$

Unfortunately, we do not know how to evaluate the integral in u, so all we have done is to equate two unknown quantities. ▲

As in Example 7, after a substitution, the integral $\int f(u)\,du$ might still be something we do not know how to evaluate. In that case it may be necessary to make another substitution or use a completely different method. There is an infinite choice of substitutions available in any given situation. It takes practice to learn to choose one that works.

In general, integration is a trial-and-error process that involves a certain amount of educated guessing. What is more, the antiderivatives of such innocent-looking functions as

$$\frac{1}{\sqrt{(1-x^2)(1-2x^2)}} \quad \text{and} \quad \frac{1}{\sqrt{3-\sin^2 x}}$$

cannot be expressed in any way as algebraic combinations and compositions of polynomials, trigonometric functions, or exponential functions. (The proof of a statement like this is not elementary; it belongs to a subject known as "differential algebra".) Despite these difficulties, you can learn to integrate many functions, but the learning process is slower than for differentiation, and practice is more important than ever.

Since integration is harder than differentiation, one often uses tables of integrals. A short table is available on the endpapers of this book, and extensive books of tables are on the market. (Two of the most popular are Burington's and the CRC tables, both of which contain a great deal of mathematical data in addition to the integrals.) Using these tables requires a knowledge of the basic integration techniques, though, and that is why you still need to learn them.

Example 8 Evaluate $\int_1^3 \frac{dx}{x\sqrt{1+x}}$ using the tables of integrals.

Solution We search the tables for a form similar to this and find number 49 with $a = 1$, $b = 1$. Thus

$$\int \frac{dx}{x\sqrt{1+x}} = \ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + C.$$

Hence

$$\int_1^3 \frac{dx}{x\sqrt{1+x}} = \ln\left|\frac{\sqrt{4}-1}{\sqrt{4}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| = \ln\frac{1}{3} - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|$$

$$= \ln\left[\frac{\sqrt{2}+1}{3(\sqrt{2}-1)}\right] = \ln\left(1 + \frac{2}{3}\sqrt{2}\right). \blacktriangle$$

Exercises for Section 7.3

Evaluate the definite integrals in Exercises 1–22.

1. $\int_{-1}^{1} \sqrt{x+2}\, dx$
2. $\int_{2}^{3} \frac{dt}{t-1}$
3. $\int_{0}^{2} x\sqrt{x^2+1}\, dx$
4. $\int_{0}^{1} t\sqrt{t^2+1}\, dt$
5. $\int_{2}^{4} (x+1)(x^2+2x+1)^{5/4}\, dx$
6. $\int_{1}^{2} \frac{\sqrt{1+\ln x}}{x}\, dx$
7. $\int_{1}^{3} \frac{3x}{(x^2+5)^2}\, dx$
8. $\int_{1}^{2} \frac{t^2+1}{\sqrt{t^3+3t+3}}\, dt$
9. $\int_{0}^{1} xe^{(x^2)}\, dx$
10. $\int_{0}^{1} \frac{e^x}{1+e^{2x}}\, dx$
11. $\int_{0}^{\pi/6} \sin(3\theta+\pi)\, d\theta$
12. $\int_{0}^{\pi} \sin(\theta/2+\pi/4)\, d\theta$
13. $\int_{-\pi/2}^{\pi/2} 5\cos^2 x \sin x\, dx.$
14. $\int_{\pi/4}^{\pi/2} \frac{\csc^2 y}{\cot^2 y + 2\cot y + 1}\, dy.$
15. $\int_{0}^{\sqrt{\pi}} x\sin(x^2)\, dx.$
16. $\int_{0}^{1} \frac{x^2}{x^3+1}\, dx.$
17. $\int_{\pi/8}^{\pi/4} \tan\theta\, d\theta.$
18. $\int_{\pi/4}^{\pi/2} \cot\theta\, d\theta.$
19. $\int_{0}^{\pi/2} \sin x \cos x\, dx.$
20. $\int_{1}^{\pi/2} [\ln(\sin x) + (x\cot x)](\sin x)^x\, dx.$
21. $\int_{1}^{3} \frac{x^3+x-1}{x^2+1}\, dx$ (simplify first).
22. $\int_{1}^{e} \frac{2\ln(x^x)+1}{x^2}\, dx.$
23. Using the result $\int_{0}^{\pi/2} \sin^2 x\, dx = \pi/4$ (See Exercise 57, Section 7.1), compute each of the following integrals: (a) $\int_{0}^{\pi} \sin^2(x/2)\, dx$; (b) $\int_{\pi/2}^{\pi} \sin^2(x-\pi/2)\, dx$; (c) $\int_{0}^{\pi/4} \cos^2(2x)\, dx.$
24. (a) By combining the shifting and scaling rules, find a formula for $\int f(ax+b)\, dx.$
    (b) Find $\int_{2}^{3} \frac{dx}{4x^2+12x+9}$ [*Hint*: Factor the denominator.]
25. What happens in the integral

$$\int_{0}^{1} \frac{(x^2+3x)}{\sqrt[3]{x^3+3x^2+1}}\, dx$$

if you make the substitution $u = x^3+3x^2+1$?

26. What becomes of the integral $\int_{0}^{\pi/2} \cos^4 x\, dx$ if you make the substitution $u = \cos x$?

Evaluate the integrals in Exercises 27–30 using the tables.

27. $\int_{0}^{1} \frac{dx}{3x^2+2x+1}$
28. $\int_{1}^{2} \frac{\sqrt{x^2-1}}{x}\, dx$
29. $\int_{0}^{1} \frac{dx}{\sqrt{3x^2+2x+1}}$
30. $\int_{2}^{3} \frac{\sqrt{x^2-2}}{x^4}\, dx$

31. Given two functions f and g, define a function h by

$$h(x) = \int_0^1 f(x - t)g(t)\, dt.$$

Show that

$$h(x) = \int_{x-1}^{x} g(x - t)f(t)\, dt.$$

32. Give another solution to Example 5 by writing $\cos^2\theta - \sin^2\theta = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$ and using the substitution $u = \cos\theta + \sin\theta$.

33. Find the area under the graph of the function $y = (x+1)/(x^2 + 2x + 2)^{3/2}$ from $x = 0$ to $x = 1$.

34. The curve $x^2/a^2 + y^2/b^2 = 1$, where a and b are positive, describes an *ellipse* (Fig. 7.3.1). Find the area of the region inside this ellipse. [*Hint*: Write half the area as an integral and then change variables in the integral so that it becomes the integral for the area inside a semicircle.]

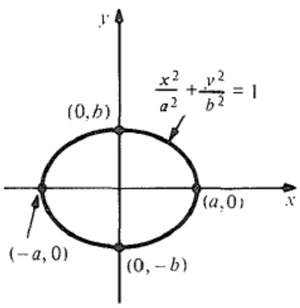

Figure 7.3.1. Find the area inside the ellipse.

35. The curve $y = x^{1/3}$, $1 \leqslant x \leqslant 8$, is revolved about the y axis to generate a surface of revolution of area s. In Chapter 10 we will prove that the area is given by $s = \int_1^2 2\pi y^3 \sqrt{1 + 9y^4}\, dy$. Evaluate this integral.

★36. Let $f(x) = \int_1^x (dt/t)$. Show, using substitution, and without using logarithms, that $f(a) + f(b) = f(ab)$ if $a, b > 0$. [*Hint*: Transform $\int_a^{ab} \frac{dt}{t}$ by a change of variables.]

37. (a) Find $\int_0^{\pi/2} \cos^2 x \sin x\, dx$ by substituting $u = \cos x$ and changing the endpoints.

★(b) Is the formula

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

valid if $a < b$, yet $g(a) > g(b)$? Discuss.