7.4 Integration by Parts

Integrating the product rule leads to the method of integration by parts.

The second of the two important new methods of integration is developed in this section. The method parallels that of substitution, with the chain rule replaced by the product rule.

The product rule for derivatives asserts that

$$(FG)'(x) = F'(x)G(x) + F(x)G'(x).$$

Since $F(x)G(x)$ is an antiderivative for $F'(x)G(x) + F(x)G'(x)$, we can write

$$\int \left[F'(x)G(x) + F(x)G'(x) \right] dx = F(x)G(x) + C.$$

Applying the sum rule and transposing one term leads to the formula

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx + C.$$

If the integral on the right-hand side can be evaluated, it will have its own constant C, so it need not be repeated. We thus write

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx, \tag{1}$$

which is the formula for integration by parts. To apply formula (1) we need to break up a given integrand as a product $F(x)G'(x)$, write down the right-hand side of formula (1), and hope that we can integrate $F'(x)G(x)$. Integrands involving trigonometric, logarithmic, and exponential functions are often good candidates for integration by parts, but practice is necessary to learn the best way to break up an integrand as a product.

Example 1 Evaluate $\int x \cos x\,dx$.

Solution If we remember that $\cos x$ is the derivative of $\sin x$, we can write $x \cos x$ as $F(x)G'(x)$, where $F(x) = x$ and $G(x) = \sin x$. Applying formula (1), we have

$$\int x \cos x\,dx = x \cdot \sin x - \int 1 \cdot \sin x\,dx = x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x + C.$$

Checking by differentiation, we have

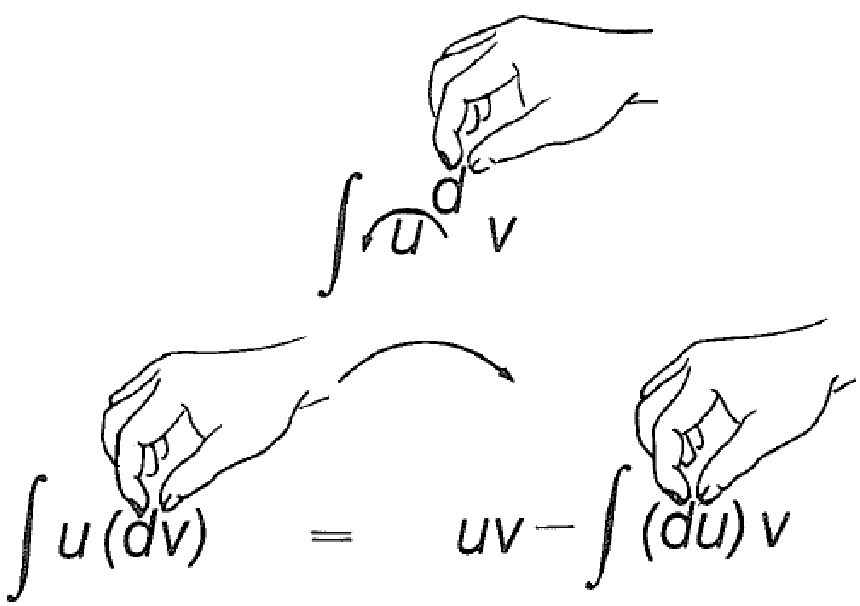
$$\frac{d}{dx}(x \sin x + \cos x) = x \cos x + \sin x - \sin x = x \cos x,$$

as required. ▲

It is often convenient to write formula (1) using differential notation. Here we write $u = F(x)$ and $v = G(x)$. Then $du/dx = F'(x)$ and $dv/dx = G'(x)$. Treating the derivatives as if they were quotients of "differentials" du, dv, and dx, we have $du = F'(x)\,dx$ and $dv = G'(x)\,dx$. Substituting these into formula (1) gives

$$\int u\,dv = uv - \int v\,du \tag{2}$$

(see Fig. 7.4.1).



Integration by Parts

To evaluate $\int h(x)\,dx$ by parts:

1. Write $h(x)$ as a product $F(x)G'(x)$, where the antiderivative $G(x)$ of $G'(x)$ is known.
2. Take the derivative $F'(x)$ of $F(x)$.
3. Use the formula

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx,$$

i.e., with $u = F(x)$ and $v = G(x)$,

$$\int u\,dv = uv - \int v\,du.$$

When you use integration by parts, to integrate a function h write $h(x)$ as a product $F(x)G'(x) = u\,dv/dx$; the factor $G'(x)$ is a function whose antiderivative $v = G(x)$ can be found. With a good choice of $u = F(x)$ and $v = G(x)$, the integral $\int F'(x)G(x)\,dx = \int v\,du$ becomes simpler than the original problem $\int u\,dv$. The ability to make good choices of u and v comes with practice. A last reminder—don't forget the minus sign.

Example 2 Find (a) $\int x \sin x\,dx$ and (b) $\int x^2 \sin x\,dx$.

Solution (a) (*Using formula* (1)) Let $F(x) = x$ and $G'(x) = \sin x$. Integrating $G'(x)$ gives $G(x) = -\cos x$; also, $F'(x) = 1$, so

$$\begin{aligned}\int x \sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x - (-\sin x) + C \\ &= -x\cos x + \sin x + C.\end{aligned}$$

(b) (*Using formula* (2)) Let $u = x^2$, $dv = \sin x\,dx$. To apply formula (2) for integration by parts, we need to know v. But $v = \int dv = \int \sin x\,dx = -\cos x$. (We leave out the arbitrary constant here and will put it in at the end of the problem.)

Now

$$\begin{aligned}\int x^2 \sin x\,dx &= uv - \int v\,du \\ &= -x^2\cos x - \int -\cos x \cdot 2x\,dx \\ &= -x^2\cos x + 2\int x\cos x\,dx.\end{aligned}$$

Using the result of Example 1, we obtain

$$-x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Check this result by differentiating—it is nice to see all the cancellation. ▲

Integration by parts is also commonly used in integrals involving e^x and $\ln x$.

Example 3 (a) Find $\int \ln x\, dx$ using integration by parts. (b) Find $\int xe^x\, dx$.

Solution (a) Here, let $u = \ln x$, $dv = 1\, dx$. Then $du = dx/x$ and $v = \int 1\, dx = x$. Applying the formula for integration by parts, we have

$$\int \ln x\, dx = uv - \int v\, du = (\ln x)x - \int x \frac{dx}{x}$$
$$= x \ln x - \int 1\, dx = x \ln x - x + C.$$

(Compare Example 7, Section 7.1.)

(b) Let $u = x$ and $v = e^x$, so $dv = e^x\, dx$. Thus, using integration by parts,

$$\int xe^x\, dx = \int u\, dv = uv - \int v\, du$$
$$= xe^x - \int e^x\, dx = xe^x - e^x + C. \blacktriangle$$

Next we consider an example involving both e^x and $\sin x$.

Example 4 Apply integration by parts twice to find $\int e^x \sin x\, dx$.

Solution Let $u = \sin x$ and $v = e^x$, so $dv = e^x\, dx$ and

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx.$$

Repeating the integration by parts,

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx, \qquad (4)$$

where, this time, $u = \cos x$ and $v = e^x$. Substituting formula (4) into (3), we get

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx.$$

The unknown integral $\int e^x \sin x\, dx$ appears twice in this equation. Writing "I" for this integral, we have

$$I = e^x \sin x - e^x \cos x - I,$$

and solving for I gives

$$I = \tfrac{1}{2} e^x (\sin x - \cos x),$$

i.e.,

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C.$$

Some students like to remember this as "the I method." $\blacktriangle$

Some special purely algebraic expressions can also be handled by a clever use of integration by parts, as in the next example.

Example 5 Find $\int x^7(x^4+1)^{2/3}\,dx$.

Solution By taking x^3 out of x^7 and grouping it with $(x^4+1)^{2/3}$, we get an expression which we can integrate. Specifically, we set $dv = 4x^3(x^4+1)^{2/3}\,dx$, leaving $u = x^4/4$. Using integration by substitution, we get $v = \frac{3}{5}(x^4+1)^{5/3}$, and differentiating, we get $du = x^3\,dx$. Hence

$$\int x^7(x^4+1)^{2/3}\,dx = \frac{3x^4}{20}(x^4+1)^{5/3} - \frac{3}{5}\int x^3(x^4+1)^{5/3}\,dx.$$

Substituting $w = (x^4+1)$ gives

$$\int x^3(x^4+1)^{5/3}\,dx = \tfrac{3}{32}(x^4+1)^{8/3} + C;$$

hence

$$\int x^7(x^4+1)^{2/3}\,dx = \tfrac{3}{20}x^4(x^4+1)^{5/3} - \tfrac{9}{160}(x^4+1)^{8/3} + C$$

$$= \tfrac{3}{160}(x^4+1)^{5/3}(5x^4-3) + C. \blacktriangle$$

Using integration by parts and then the fundamental theorem of calculus, we can calculate definite integrals.

Example 6 Find $\int_{-\pi/2}^{\pi/2} x\sin x\,dx$.

Solution From Example 2 (a) we have $\int x\sin x\,dx = -x\cos x + \sin x + C$, so

$$\int_{-\pi/2}^{\pi/2} x\sin x\,dx = (-x\cos x + \sin x)\Big|_{-\pi/2}^{\pi/2}$$

$$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left[\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= (0+1) - \left[0 + (-1)\right] = 2. \blacktriangle$$

Example 7 Find (a) $\int_0^{\ln 2} e^x \ln(e^x + 1)\, dx$ and (b) $\int_1^e \sin(\ln x)\, dx$.

Solution (a) Notice that e^x is the derivative of $(e^x + 1)$, so we first make the substitution $t = e^x + 1$. Then

$$\int_0^{\ln 2} e^x \ln(e^x + 1)\, dx = \int_2^3 \ln t\, dt,$$

and, from Example 3, $\int \ln t\, dt = t \ln t - t + C$. Therefore

$$\int_0^{\ln 2} e^x \ln(e^x + 1)\, dx = (t \ln t - t)\Big|_2^3 = (3\ln 3 - 3) - (2\ln 2 - 2)$$
$$= 3\ln 3 - 2\ln 2 - 1 \approx 0.9095.$$

(b) Again we begin with a substitution. Let $u = \ln x$, so that $x = e^u$ and $du = (1/x)\,dx$. Then $\int \sin(\ln x)\, dx = \int (\sin u)e^u\, du$, which was evaluated in Example 4. Hence

$$\int_1^e \sin(\ln x)\, dx = \int_0^1 e^u \sin u\, du = \frac{1}{2} e^u(\sin u - \cos u)\Big|_0^1$$
$$= \left[\frac{1}{2} e^1(\sin 1 - \cos 1)\right] - \left[\frac{1}{2} e^0(\sin 0 - \cos 0)\right]$$
$$= \frac{e}{2}\left(\sin 1 - \cos 1 + \frac{1}{e}\right). \blacktriangle$$

Example 8 Find the area under the nth bend of $y = x \sin x$ in the first quadrant (see Fig. 7.4.2).

Solution The nth bend occurs between $x = (2n-2)\pi$ and $(2n-1)\pi$. (Check $n = 1$ and $n = 2$ with the figure.) The area under this bend can be evaluated using integration by parts [Example 2(a)]:

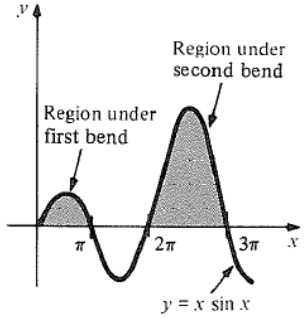


Figure 7.4.2. What is the area under the nth bend?

$$\begin{aligned}\int_{(2n-2)\pi}^{(2n-1)\pi} x \sin x\, dx &= -x \cos x + \sin x \Big|_{(2n-2)\pi}^{(2n-1)\pi} \\ &= -(2n-1)\pi \cos\left[(2n-1)\pi\right] + \sin\left[(2n-1)\pi\right] \\ &\quad + (2n-2)\pi \cos(2n-2)\pi - \sin(2n-2)\pi \\ &= -(2n-1)\pi(-1) + 0 + (2n-2)\pi(1) - 0 \\ &= (2n-1)\pi + (2n-2)\pi = (4n-3)\pi.\end{aligned}$$

Thus the areas under successive bends are π, 5π, 9π, 13π, and so forth. ▲

We shall now use integration by parts to obtain a formula for the integral of the inverse of a function.

If f is a differentiable function, we write $f(x) = 1 \cdot f(x)$; then

$$\int f(x)\, dx = \int 1 \cdot f(x)\, dx = xf(x) - \int xf'(x)\, dx. \tag{5}$$

Introducing $y = f(x)$ as a new variable, with $dx = dy/f'(x)$, we get

$$\int f(x)\, dx = xy - \int x\, dy. \tag{6}$$

Assuming that f has an inverse function g, we have $x = g(y)$, and equation (6) becomes

$$\int f(x)\, dx = xf(x) - \int g(y)\, dy. \tag{7}$$

Thus we can integrate f if we know how to integrate its inverse. In the notation $y = f(x)$, equation (7) becomes

$$\int y\, dx = xy - \int x\, dy. \tag{8}$$

Notice that equation (8) looks just like the formula for integration by parts, but we are now considering x and y as functions of one another rather than as two functions of a third variable.

Example 9 Use equation (8) to compute $\int \ln x \, dx$.

Solution Viewing $y = \ln x$ as the inverse function of $x = e^y$, equation (8) reads

$$\int \ln x \, dx = xy - \int e^y \, dy = x \ln x - e^y + C = x \ln x - x + C,$$

which is the same result (and essentially the same method) as in Example 3. ▲

We can also state our result in terms of antiderivatives. If $G(y)$ is an antiderivative for $g(y)$, then

$$F(x) = xf(x) - G(f(x)) \tag{9}$$

is an antiderivative for f. (This can be checked by differentiation.)

Example 10 (a) Find an antiderivative for $\cos^{-1}x$. (b) Find $\int \csc^{-1}\sqrt{x} \, dx$.

Solution (a) If $f(x) = \cos^{-1}x$, then $g(y) = \cos y$ and $G(y) = \sin y$. By formula (9),

$$F(x) = x\cos^{-1}x - \sin(\cos^{-1}x);$$

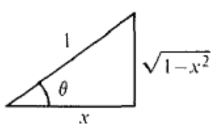


Figure 7.4.3. $\sin(\cos^{-1}x) = \sqrt{1-x^2}$

But $\sin(\cos^{-1}x) = \sqrt{1-x^2}$ (Fig. 7.4.3), so

$$F(x) = x\cos^{-1}x - \sqrt{1-x^2}$$

is an antiderivative for $\cos^{-1}x$. This may be checked by differentiation.

(b) If $y = \csc^{-1}\sqrt{x}$, we have $\csc y = \sqrt{x}$ and $x = \csc^2 y$. Then

$$\begin{aligned}\int \csc^{-1}\sqrt{x} \, dx &= \int y \, dx = xy - \int x \, dy \\ &= x\csc^{-1}\sqrt{x} - \int \csc^2 y \, dy \\ &= x\csc^{-1}\sqrt{x} + \cot y + C \\ &= x\csc^{-1}\sqrt{x} + \cot(\csc^{-1}\sqrt{x}) + C \\ &= x\csc^{-1}\sqrt{x} + \sqrt{x-1} + C \qquad \text{(see Fig. 7.4.4).}\end{aligned}$$

▲

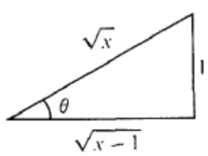


Figure 7.4.4. $\theta = \csc^{-1}\sqrt{x}$.

Example 11 (a) Find $\int \sqrt{\sqrt{x}+1}\,dx$. (b) Find $\int x\cos^{-1}x\,dx$, $0<x<1$.

Solution (a) If $y=\sqrt{\sqrt{x}+1}$, then $y^2=\sqrt{x}+1$, $\sqrt{x}=y^2-1$, and $x=(y^2-1)^2$. Thus we have

$$\int \sqrt{\sqrt{x}+1}\,dx = xy-\int x\,dy = x\sqrt{\sqrt{x}+1}-\int (y^4-2y^2+1)\,dy$$

$$= x\sqrt{\sqrt{x}+1}-\frac{1}{5}y^5+\frac{2}{3}y^3-y+C$$

$$= x\sqrt{\sqrt{x}+1}-\frac{1}{5}(\sqrt{x}+1)^{5/2}+\frac{2}{3}(\sqrt{x}+1)^{3/2}$$

$$-(\sqrt{x}+1)^{1/2}+C.$$

(b) Integrating by parts,

$$\int x\cos^{-1}x\,dx=\frac{x^2}{2}\cos^{-1}x+\int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx.$$

The last integral may be evaluated by letting $x=\cos u$:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx=-\int \frac{\cos^2 u}{\sin u}\sin u\,du=-\int \cos^2 u\,du$$

But $\cos^2 u=\dfrac{\cos 2u+1}{2}$, so

$$\int \cos^2 u\,du=\frac{1}{4}\sin 2u+\frac{u}{2}+C=\frac{1}{2}\sin u\cos u+\frac{u}{2}+C.$$

Thus,

$$\int x\cos^{-1}x\,dx=\frac{x^2}{2}\cos^{-1}x-\frac{1}{4}\sin(\cos^{-1}x)x-\frac{\cos^{-1}x}{4}+C$$

$$=\frac{x^2}{2}\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}-\frac{1}{4}\cos^{-1}x+C. \blacktriangle$$

Exercises for Section 7.4

Evaluate the indefinite integrals in Exercises 1–26 using integration by parts.

1. $\int (x+1)\cos x\,dx$
2. $\int (x-2)\sin x\,dx$
3. $\int x\cos(5x)\,dx$
4. $\int x\sin(10x)\,dx$
5. $\int x^2\cos x\,dx$
6. $\int x^2\sin x\,dx$
7. $\int (x+2)e^x\,dx$
8. $\int (x^2-1)e^{2x}\,dx$
9. $\int \ln(10x)\,dx$
10. $\int x\ln x\,dx$
11. $\int x^2\ln x\,dx$
12. $\int \ln(9+x^2)\,dx$
13. $\int s^2e^{3s}\,ds$
14. $\int (s+1)^2e^s\,ds.$
15. $\int \frac{x^5}{(x^3-4)^{2/3}}\,dx$
16. $\int \frac{x^2}{(x^2+1)^2}\,dx$
17. $\int 2t^3\cos t^2\,dt$
18. $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$
19. $\int \frac{1}{x^3}\cos\frac{1}{x}\,dx$
20. $\int x\sin(\ln x)\,dx$
21. $\int \tan x\ln(\cos x)\,dx$
22. $\int e^{2x}e^{(e^x)}\,dx$
23. $\int \cos^{-1}(2x)\,dx$
24. $\int \sin^{-1}x\,dx$
25. $\int \sqrt{\frac{1}{y}-1}\;dy$
26. $\int (\sqrt{x}-2)^{1/5}\,dx$
27. Find $\int \sin x\cos x\,dx$ by using integration by parts with $u=\sin x$ and $dv=\cos x\,dx$. Compare the result with substituting $u=\sin x$.
28. Compute $\int \sqrt{x}\,dx$ by the rule for inverse functions. Compare with the result given by the power rule.
29. What happens in Example 2(a) if you choose $F'(x)=x$ and $G(x)=\sin x$?
30. What would have happened in Example 5, if in the integral $\int e^x\cos x\,dx$ obtained in the first integral by parts, you had taken $u=e^x$ and $v=\sin x$ and integrated by parts a second time?

Evaluate the definite integrals in Exercises 31–46.

31. $\int_0^{\pi/5}(8+5\theta)(\sin 5\theta)\,d\theta$
32. $\int_1^2 x\ln x\,dx$
33. $\int_1^3 \ln x^3\,dx$
34. $\int_0^1 xe^x\,dx$
35. $\int_0^{\pi/4}(x^2+x-1)\cos x\,dx$
36. $\int_0^{\pi/2}\sin 3x\cos 2x\,dx$
37. $\int_{1/8}^{1/4}\cos^{-1}(4x)\,dx$
38. $\int_0^1 x\tan^{-1}x\,dx$
39. $\int_1^e (\ln x)^2\,dx$
40. $\int_0^{\pi/2}\sin 2x\cos x\,dx.$
41. $\int_{-\pi}^{\pi} e^{2x}\sin(2x)\,dx.$
42. $\int_0^{\pi^2}\sin\sqrt{x}\,dx.$ [*Hint*: Change variables first.]
43. $\int_1^2 x^{1/3}(x^{2/3}+1)^{3/2}\,dx.$
44. $\int_0^1 \frac{x^3}{(x^2+1)^{1/2}}\,dx.$
45. $\int_0^{1/2\sqrt{2}}\sin^{-1}2x\,dx.$
46. $\int_0^1 \cos^{-1}(\sqrt{y})\,dy.$

47. Show that

$$\int_0^1 \sqrt{2-x^2}\,dx - \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = (1-\pi/2)/2.$$

48. Find $\int_2^{34} f(x)\,dx$, where f is the inverse function of $g(y) = y^5 + y$.

49. Find $\int_0^{2\pi} x \sin ax\,dx$ as a function of a. What happens to this integral as a becomes larger and larger? Can you explain why?

50. (a) Integrating by parts twice (see Example 4), find $\int \sin ax \cos bx\,dx$, where $a^2 \neq b^2$.
   (b) Using the formula $\sin 2x = 2\sin x \cos x$, find $\int \sin ax \cos bx\,dx$ when $a = \pm b$.
   (c) Let $g(a) = (4/\pi)\int_0^{\pi/2} \sin x \sin ax\,dx$. Find a formula for $g(a)$. (The formula will have to distinguish the cases $a^2 \neq 1$ and $a^2 = 1$.)
   (d) Evaluate $g(a)$ for $a = 0.9$, 0.99, 0.999, 0.9999, and so on. Compare the results with $g(1)$. Also try $a = 1.1$, 1.01, 1.001, and so on. What do you guess is true about the function g at $a = 1$?

51. (a) Integrating by parts twice, show that

$$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b\sin bx + a\cos bx}{a^2+b^2}\right) + C.$$

   (b) Evaluate $\int_0^{\pi/10} e^{3x}\cos 5x\,dx$.

52. (a) Prove the following *reduction formula*:

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx.$$

   (b) Evaluate $\int_0^3 x^3 e^x\,dx$

53. (a) Prove the following *reduction formula*:

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

   (b) Use part (a) to show that

$$\int \cos^2 x\,dx = \frac{1}{2}(\cos x \sin x + x) + C$$

and

$$\int \cos^4 x\,dx = \frac{1}{4}\left(\cos^3 x \sin x + \frac{3}{2}\cos x \sin x + \frac{3x}{2}\right) + C.$$

54. The mass density of a beam is $\rho = x^2 e^{-x}$ kilograms per centimeter. The beam is 200 centimeters long, so its mass is $M = \int_0^{200}\rho\,dx$ kilograms. Find the value of M.

55. The volume of the solid formed by rotation of the plane region enclosed by $y = 0$, $y = \sin x$, $x = 0$, $x = \pi$, around the y axis, will be shown in Chapter 9 to be given by $V = \int_0^{\pi} 2\pi x \sin x\,dx$. Find V.

56. The *Fourier series* analysis of the sawtooth wave requires the computation of the integral

$$b_m = \frac{\omega^2 A}{2\pi^2}\int_{-\pi/\omega}^{\pi/\omega} t\sin(m\omega t)\,dt,$$

where m is an integer and ω and A are nonzero constants. Compute it.

57. The current i in an underdamped RLC circuit is given by

$$i = EC\left(\frac{\alpha^2}{\omega} + \omega\right)e^{-\alpha t}\sin(\omega t).$$

The constants are E = constant emf, switched on at $t = 0$, C = capacitance in farads, R = resistance in ohms, L = inductance in henrys, $\alpha = R/2L$, $\omega = (1/2L)(4L/C - R^2)^{1/2}$.
   (a) The charge Q in coulombs is given by $dQ/dt = i$, and $Q(0) = 0$. Find an integral formula for Q, using the fundamental theorem of calculus.
   (b) Determine Q by integration.

58. A critically damped RLC circuit with a steady emf of E volts has current $i = EC\alpha^2 te^{-\alpha t}$, where $\alpha = R/2L$. The constants R, L, C are in ohms, henrys, and farads, respectively. The charge Q in coulombs is given by $Q(T) = \int_0^T i\,dt$. Find it explicitly, using integration by parts.

★59. Draw a figure to illustrate the formula for integration of inverse functions:

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy,$$

where $0 < a < b$, $0 < f(a) < f(b)$, f is increasing on $[a, b]$, and g is the inverse function of f.

★60. (a) Suppose that $\phi'(x) > 0$ for all x in $[0, \infty)$ and $\phi(0) = 0$. Show that if $a \geqslant 0$, $b \geqslant 0$, and b is in the domain of ϕ^{-1}, then *Young's inequality* holds:

$$ab \leqslant \int_0^a \phi(x)\,dx + \int_0^b \phi^{-1}(y)\,dy,$$

where ϕ^{-1} is the inverse function to ϕ. [*Hint*: Express $\int_0^b \phi^{-1}(y)\,dy$ in terms of an integral of ϕ by using the formula for integrating an inverse function. Consider separately the cases $\phi(a) \leqslant b$ and $\phi(a) \geqslant b$. For the latter, prove the inequality $\int_{\phi^{-1}(b)}^a \phi(x)\,dx \geqslant \int_{\phi^{-1}(b)}^a b\,dx = b[a - \phi^{-1}(b)]$.]
   (b) Prove (a) by a geometric argument based on Exercise 59.
   (c) Using the result of part (a), show that if $a, b \geqslant 0$ and $p, q > 1$, with $1/p + 1/q = 1$, then *Minkowski's inequality* holds:

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

★61. If f is a function on $[0, 2\pi]$, the numbers

$$a_n = (1/\pi)\int_0^{2\pi} f(x)\cos nx\,dx,$$

$$b_n = (1/\pi)\int_0^{2\pi} f(x)\sin nx\,dx$$

are called the *Fourier coefficients* of f ($n = 0, \pm 1, \pm 2, \ldots$). Find the Fourier coefficients of: (a) $f(x) = 1$; (b) $f(x) = x$; (c) $f(x) = x^2$; (d) $f(x) = \sin 2x + \sin 3x + \cos 4x$.

★62. Following Example 5, find a general formula for $\int x^{2n-1}(x^n + 1)^m\,dx$, where n and m are rational numbers with $n \neq 0$, $m \neq -1, -2$.

Review Exercises for Chapter 7

Evaluate the integrals in Exercises 1–46.

1. $\int (x + \sin x)\, dx$
2. $\int \left(x + \dfrac{1}{\sqrt{1-x^2}}\right) dx$
3. $\int (x^3 + \cos x)\, dx$
4. $\int (8t^4 - 5\cos t)\, dt$
5. $\int \left(e^x - x^2 - \dfrac{1}{x} + \cos x\right) dx$
6. $\int \left(3^x - \dfrac{3}{x} + \cos x\right) dx$
7. $\int (e^\theta + \theta^2)\, d\theta$
8. $\int \dfrac{\sqrt[3]{x^2} - x^{5/2}}{\sqrt{x}}\, dx$
9. $\int x^2 \sin x^3\, dx$
10. $\int \tan x \sec^2 x\, dx$
11. $\int x^2 e^{(x^3)}\, dx$
12. $\int x e^{(x^2)}\, dx$
13. $\int (x+2)^5\, dx$
14. $\int \dfrac{dx}{3x+4}$
15. $\int x^2 e^{(4x^3)}\, dx$
16. $\int (1 + 3x^2)\exp(x + x^3)\, dx$
17. $\int 2\cos^2 2x \sin 2x\, dx$
18. $\int 3 \sin 3x \cos 3x\, dx$
19. $\int x \tan^{-1} x\, dx$
20. $\int x\sqrt{5 - x^2}\, dx$
21. $\int \left[\dfrac{1}{\sqrt{4-t^2}} + t^2\right] dt$
22. $\int \dfrac{e^{2x}}{1 + e^{4x}}\, dx$
23. $\int x e^{4x}\, dx$
24. $\int x e^{6x}\, dx$
25. $\int x^2 \cos x\, dx$
26. $\int x^2 e^{2x}\, dx$
27. $\int e^{-x} \cos x\, dx$
28. $\int e^{2x} \tan e^{2x}\, dx$
29. $\int x^2 \ln 3x\, dx$
30. $\int x^3 \ln x\, dx$
31. $\int x\sqrt{x+3}\, dx$
32. $\int x^2\sqrt{x+1}\, dx$
33. $\int x \cos 3x\, dx$
34. $\int t \cos 2t\, dt$
35. $\int 3x \cos 2x\, dx$
36. $\int \sin 2x \cos x\, dx$
37. $\int x^3 e^{(x^2)}\, dx$
38. $\int x^5 e^{(x^3)}\, dx$
39. $\int x(\ln x)^2\, dx$
40. $\int (\ln x)^2\, dx$
41. $\int e^{\sqrt{x}}\, dx$
42. $\int \dfrac{dx}{x^2 + 2x + 3}$ (Complete the square.)
43. $\int [\cos x]\ln(\sin x)\, dx$
44. $\int \dfrac{\ln\sqrt{x}}{\sqrt{x}}\, dx$
45. $\int \tan^{-1} x\, dx$
46. $\int \cos^{-1}(12x)\, dx$

Evaluate the definite integrals in Exercises 47–58.

47. $\int_{-1}^{0} x e^{-x}\, dx$
48. $\int_1^e x \ln(5x)\, dx$
49. $\int_0^{\pi/5} x \sin 5x\, dx$
50. $\int_0^{\pi/4} x \cos 2x\, dx$
51. $\int_1^2 x^{-2} \cos(1/x)\, dx$
52. $\int_0^{\pi/2} x^2 \cos(x^3) \sin(x^3)\, dx$
53. $\int_0^{\pi/4} x \tan^{-1} x\, dx$
54. $\int_1^{\ln(\pi/4)} e^x \tan e^x\, dx$
55. $\int_{a+1}^{a+2} \dfrac{t}{\sqrt{t-a}}\, dt$ (substitute $x = \sqrt{t-a}$)
56. $\int_0^1 \dfrac{\sqrt{x}}{x+1}\, dx$
57. $\int_0^1 x\sqrt{2x+3}\, dx$
58. $\int_0^{\sqrt{3}} \dfrac{3}{3 + u^2}\, du$

In Exercises 59–66, sketch the region under the graph the given function on the given interval and find its area.

59. $40 - x^3$ on $[0, 3]$
60. $\sin x + 2x$ on$[0, 4\pi]$

61. $3x/\sqrt{x^2+9}$ on $[0, 4]$
62. $x\sin^{-1}x + 2$ on $[0, 1]$
63. $\sin x$ on $[0, \pi/4]$
64. $\sin 2x$ on $[0, \pi/2]$
65. $1/x$ on $[2, 4]$
66. xe^{-2x} on $[0, 1]$

67. Let R_n be the region bounded by the x axis, the line $x = 1$, and the curve $y = x^n$. The area of R_n is what fraction of the area of the triangle R_1?
68. Find the area under the graph of $f(x) = x/\sqrt{x^2+2}$ from $x = 0$ to $x = 2$.
69. Find the area between the graphs of $y = -x^3 - 2x - 6$ and $y = e^x + \cos x$ from $x = 0$ to $x = \pi/2$.
70. Find the area above the n^{th} bend of $y = x\sin x$ which lies below the x axis. (See Fig. 7.4.2).
71. Water is flowing into a tank with a rate of $10(t^2 + \sin t)$ liters per minute after time t. Calculate: (a) the number of liters stored after 30 minutes, starting at $t = 0$; (b) the average flow rate in liters per minute over this 30-minute interval.
72. The velocity of a train fluctuates according to the formula $v = (100 + e^{-3t}\sin 2\pi t)$ kilometers per hour. How far does the train travel: (a) between $t = 0$ and $t = 1$?; (b) between $t = 100$ and $t = 101$?
73. Evaluate $\int \sin(\pi x/2)\cos(\pi x)\,dx$ by integrating by parts two different ways and comparing the results.
74. Do Exercise 73 using the product formulas for sine and cosine.
75. Evaluate $\int \sqrt{(1+x)/(1-x)}\,dx$. [*Hint*: Multiply numerator and denominator by $\sqrt{1+x}$.]
76. Substitute $x = \sin u$ to evaluate

$$\int \frac{x\,dx}{\sqrt{1-x^2}}$$

and

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}}; \qquad 0 < x < 1.$$

77. Evaluate:

(a) $\int \frac{\ln x}{x}\,dx$,

(b) $\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}}$, (use $x = 3\tan u$).

78. (a) Prove the following reduction formula:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

if $n \geqslant 2$, by integration by parts, with $u = \sin^{n-1}x$, $v = -\cos x$.

(b) Evaluate $\int \sin^2 x\,dx$ by using this formula.

(c) Evaluate $\int \sin^4 x\,dx$.

79. Find $\int x^n \ln x\,dx$ using $\ln x = (1/(n+1))\ln x^{n+1}$ and the substitution $u = x^{n+1}$.
80. (a) Show that:

$$\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$$

(b) Evaluate $\int_1^2 x^2(\ln x)^2\,dx$.

81. The charge Q in coulombs for an RC circuit with sinusoidal switching satisfies the equation

$$\frac{dQ}{dt} + \frac{1}{0.04}Q = 100\sin\left(\frac{\pi}{2-5t}\right), \quad Q(0) = 0.$$

The solution is

$$Q(t) = 100e^{-25t}\int_0^t e^{25x}\cos 5x\,dx.$$

(a) Find Q explicitly by means of integration by parts.

(b) Verify that $Q(1.01) = 0.548$ coulomb. [*Hint*: Be sure to use radians throughout the calculation.]

82. What happens if $\int f(x)\,dx$ is integrated by parts with $u = f(x)$, $v = x$?
★83. Arthur Perverse believes that the product rule for integrals ought to be that $\int f(x)g(x)\,dx$ equals $f(x)\int g(x)\,dx + g(x)\int f(x)\,dx$. We wish to show him that this is not a good rule.

(a) Show that if the functions $f(x) = x^m$ and $g(x) = x^n$ satisfy Perverse's rule, then for fixed n the number m must satisfy a certain quadratic equation (assume $n, m \geqslant 0$).

(b) Show that the quadratic equation of part (a) does not have any real roots for any $n \geqslant 0$.

(c) Are there *any* pairs of functions, f and g, which satisfy Perverse's rule? (Don't count the case where one function is zero.)

★84. Derive an integration formula obtained by reading the quotient rule for derivatives backwards.
★85. Find $\int xe^{ax}\cos(bx)\,dx$.