2 Integration By Parts

We can think of integration by substitution as the counterpart of the product rule for differentiation. Suppose that $u(x)$ and $v(x)$ are continuously differentiable functions. Integration by parts is given by the following formulas:

Indefinite Integral Version:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Definite Integral Version:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)\,dx.$$

2.2 Example Problems

Problem 1. ($\star$) Find

$$\int xe^x\,dx.$$

Solution 1.

Step 1: Draw the table

$\pm$	D	I
$+$	x	e^x
$-$	1	e^x
$+\int$	0	e^x

Step 2: From the table, we have

$$\int xe^x\,dx = xe^x - e^x + C.$$

Problem 2. ($\star\star$) Find

$$\int x^6 e^x \, dx.$$

Solution 2.

Step 1: Draw the table

$\pm$	D	I
$+$	x^6	e^x
$-$	$6x^5$	e^x
$+$	$30x^4$	e^x
$-$	$120x^3$	e^x
$+$	$360x^2$	e^x
$-$	$720x$	e^x
$+$	720	e^x
$-\int$	0	e^x

Step 2: From the table, we have

$$\int x^6 e^x \, dx = x^6 e^x - 6x^5 e^x + 30x^4 e^x - 120x^2 e^x - 720xe^x + 720e^x + C.$$

Problem 3. ($\star\star$) Find

$$\int x^4 \sin x \, dx.$$

Solution 3.

Step 1: Draw the table

$\pm$	D	I
$+$	x^4	$\sin x$
$-$	$4x^3$	$-\cos x$
$+$	$12x^2$	$-\sin x$
$-$	$24x$	$\cos x$
$+$	24	$\sin x$
$-\int$	0	$-\cos x$

Step 2: From the table, we have

$$\int x^4 \sin x \, dx = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

Problem 4. (⋆⋆) Find

$$\int e^x \sin x \, dx.$$

Solution 4.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+\int$	$-\sin x$	e^x

Step 2: From the table, we have

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + D.$$

Moving all the $\int e^x \sin x \, dx$ to one side and simplifying, we can conclude

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + D \implies \int e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C.$$

Problem 5. ($\star\star\star$) Find

$$\int xe^x \cos(x)\, dx.$$

Solution 5.

Step 1: Draw the table

$\pm$	D	I
$+$	$x\cos x$	e^x
$-$	$\cos x - x\sin x$	e^x
$+\int$	$-2\sin x - x\cos x$	e^x

Step 2: From the table, we have

$$\int xe^x \cos x\, dx = xe^x \cos x - e^x \cos x + xe^x \sin x - 2\int e^x \sin x\, dx - \int xe^x \cos x\, dx.$$

Moving all the $\int xe^x \cos x\, dx$ to one side and simplifying, we can conclude

$$\begin{aligned} 2\int xe^x \cos x\, dx &= xe^x \cos x - e^x \cos x + xe^x \sin x - 2\int e^x \sin x\, dx \\ &= xe^x \cos x - e^x \cos x + xe^x \sin x - e^x \sin x + e^x \cos x + C. \qquad \text{Problem 4} \end{aligned}$$

Dividing both sides by 2, we can conclude

$$\int xe^x \cos x\, dx = \frac{1}{2}\Big(xe^x \cos x + xe^x \sin x - e^x \sin x\Big) + C.$$

Problem 6. ($\star$) Find

$$\int \ln(x)\, dx.$$

Solution 6.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln(x)$	1
$-\int$	$\frac{1}{x}$	x

Step 2: From the table, we have

$$\int \ln(x)\, dx = x\ln(x) - \int 1\, dx = x\ln(x) - x + C.$$

Problem 7. ($\star\star$) Evaluate

$$\int_1^2 x^3 \ln x\, dx.$$

Solution 7.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln x$	x^3
$-\int$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Step 2: From the table, we have

$$\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

Step 3: We can now use the fundamental theorem of calculus to compute the definite integral,

$$\int_1^2 x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\Big|_{x=1}^{x=2} = 4\ln 2 - 1 + \frac{1}{16} = 4\ln 2 - \frac{15}{16}.$$

Problem 8. ($\star\star\star$) Derive the reduction formula

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

Solution 8.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin^{n-1}(x)$	$\sin(x)$
$-\int$	$(n-1)\cos(x)\sin^{n-2}(x)$	$-\cos(x)$

Step 2: From the table, we have

$$\begin{aligned}\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x) \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x) \qquad \sin^2(x)+\cos^2(x)=1 \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx\end{aligned}$$

Moving all the the $\int \sin^n(x)\,dx$ terms to one side, we can conclude

$$\begin{aligned} n\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx \\ \Rightarrow \int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.\end{aligned}$$

Problem 9. ($\star\star$) For $x \in \mathbb{R}$, the *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Find

$$\int \operatorname{erf}(x)\, dx.$$

Solution 9. We can integrate by parts,

$\pm$	D	I
$+$	$\operatorname{erf}(x)$	1
$-\int$	$\frac{d}{dx}\operatorname{erf}(x)$	x

Since $\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ by the fundamental theorem, we have

$$\int \operatorname{erf}(x)\, dx = x\operatorname{erf}(x) - \int \frac{2x}{\sqrt{\pi}} e^{-x^2}\, dx.$$

The second integral can be solved using the substitution $u = -x^2$, $du = -2x dx$ which gives us

$$\int \operatorname{erf}(x)\, dx = x\operatorname{erf}(x) + \int \frac{1}{\sqrt{\pi}} e^u\, du = x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}} \cdot e^{-x^2} + C.$$

Remark: It is easy to check that the $x\operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C$ is an antiderivative by simply differentiating.