10.1 Curves Defined by Parametric Equations

Consider a pair of continuous functions: $f(t)$ and $g(t)$, both defined on the interval I. Form the ordered pair: $P_t = (f(t), g(t))$ for each $t \in I$.

Then as t ranges over I, P_t traces out a curve $\mathscr{C}$ in the coordinate plane.

The equations:

$$x = f(t), \quad y = g(t) \quad \text{for } t \in I$$

are called parametric equations for $\mathscr{C}$ and t is called the parameter.

Often, t represents the variable time in applications.

Examples

1 An object moves so that at time t it has the coordinates:

$$x = f(t) = t+1, \qquad y = g(t) = 2t-1 \quad \text{for} \quad 0 \le t \le 10 \text{ seconds}.$$

Describe the path of motion.

In this case, we may easily eliminate the parameter t to find the xy-equation of the path:

Since

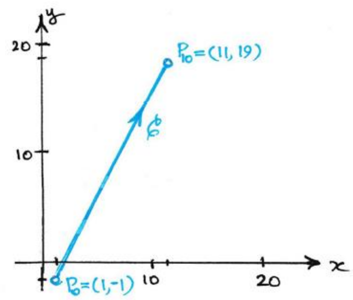
$$t = f(t) - 1 \text{ then } g(t) = 2t - 1 = 2[f(t) - 1] - 1$$

$$\Rightarrow g(t) = 2 \cdot f(t) - 3.$$

So the path $\mathscr{C}$ is along the line:

$y = 2x - 3$ from $P_0 = (f(0), g(0)) = (1, -1)$

to $P_{10} = (f(10), g(10)) = (11, 19)$

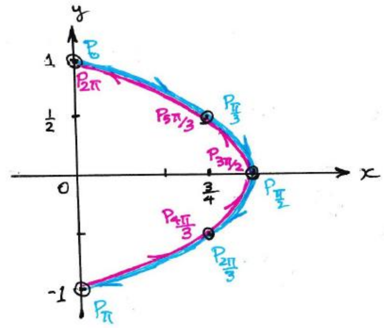


121.

2. An object moves so that at time t it has the coordinates :

$$x = f(t) = \sin^2 t \quad \text{and} \quad y = g(t) = \cos t \quad \text{for all } t \geq 0.$$

Describe the path of motion.



Note that $f(t) = \sin^2 t = 1 - \cos^2 t = 1 - (g(t))^2$.

So $x = 1 - y^2$

and the path of motion is along that portion of the parabolic arc:

$x = 1 - y^2$ in the 1^{st} and 4^{th} quadrants.

The initial point is $P_0 = (0, 1)$ and the period is 2π time units / cycle.

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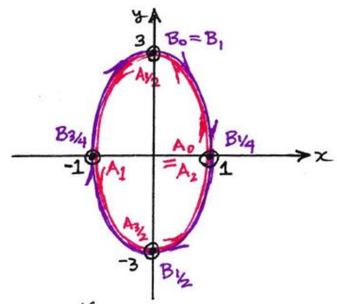
3. Describe the motions of particles A and B for $t \geq 0$ given their position equations.

Particle	Position Eq's	
A	$x = \cos(\pi t)$ $y = 3\sin(\pi t)$	A moves counter clockwise ↺ from $A_0 = (1,0)$ with period 2.
B	$x = \sin(2\pi t)$ $y = 3\cos(2\pi t)$	B moves clockwise ↻ from $B_0 = (0,3)$ with period 1.



Both paths of motion are such that:

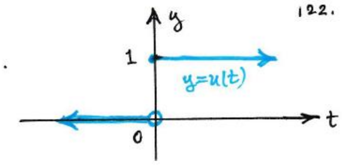
$$x^2 + \frac{y^2}{9} = \cos^2\theta + \sin^2\theta = 1$$

for $\theta = \pi t$ or $\theta = 2\pi t$. Hence, both particles traverse the same ellipse:

$$x^2 + \frac{y^2}{9} = 1.$$

122.

4. Let $u(t)$ denote the unit step function: $u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$.



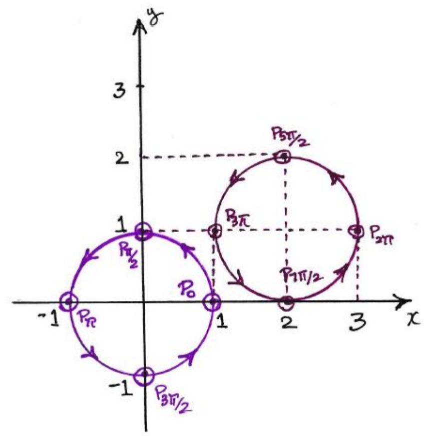
Plot the parametric curve defined by:

$x(t) = \cos t + 2\cdot u(t-2\pi)$ and $y(t) = \sin t + u(t-2\pi)$

for $0 \leq t < 4\pi$.

If $0 \leq t < 2\pi$ then $x(t) = \cos t$ and $y(t) = \sin t$ and since $\cos^2 t + \sin^2 t = 1$ for all $t \in [0, 2\pi)$ it follows that the motion is along the unit circle $x^2 + y^2 = 1$ centered at O for $t \in [0, 2\pi)$.

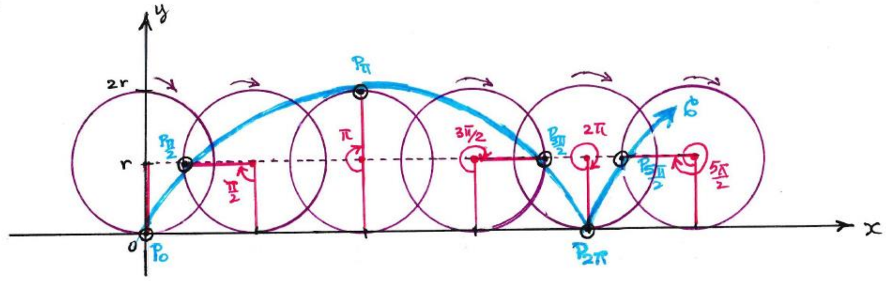
Moreover, $P_0 = (x(0), y(0)) = (1,0) = \lim\limits_{t \to 2\pi^-} (x(t), y(t)) = \lim\limits_{t \to 2\pi^-} P_t$.



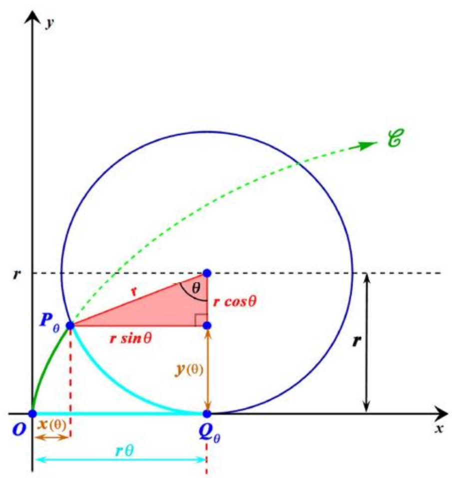
If $2\pi \leq t < 4\pi$ then $x(t) = \cos t + 2$ and $y(t) = \sin t + 1$. Since $\cos^2 t + \sin^2 t = 1$ for all $t \in [2\pi, 4\pi)$ it follows that the motion is along the unit circle: $(x-2)^2 + (y-1)^2 = 1$ centered at $(2,1)$ for $t \in [2\pi, 4\pi)$.

Note that $P_{2\pi} = (x(2\pi), y(2\pi)) = (3,1) = \lim\limits_{t \to 4\pi^-} (x(t), y(t)) = \lim\limits_{t \to 4\pi^-} P_t$.

5. Determine parametric equations to describe the curve traced by a point P located along the rim of a wheel of radius r which rolls along a straight and level path. This curve, $\mathscr{C}$, is called a cycloid. 123.



Let the parameter be θ = central angle of rotation of wheel.



Observe that when the wheel has rotated through the angle θ then it has traveled

$$|\text{arc}(\widehat{Q_\theta P_\theta})| = |\overline{OQ_\theta}| \text{ units}$$

(since when $\theta = 0$ then $P_\theta = (0,0)$).

Thus, $|OQ_\theta| = |\widehat{Q_\theta P_\theta}| = \frac{\theta}{2\pi} \cdot 2\pi r = r\theta$.

Therefore, the parametric equations for $P_\theta = (x(\theta), y(\theta))$ are:

$$\boxed{x(\theta) = r\theta - r\sin\theta \quad \text{and} \quad y(\theta) = r - r\cos\theta.}$$