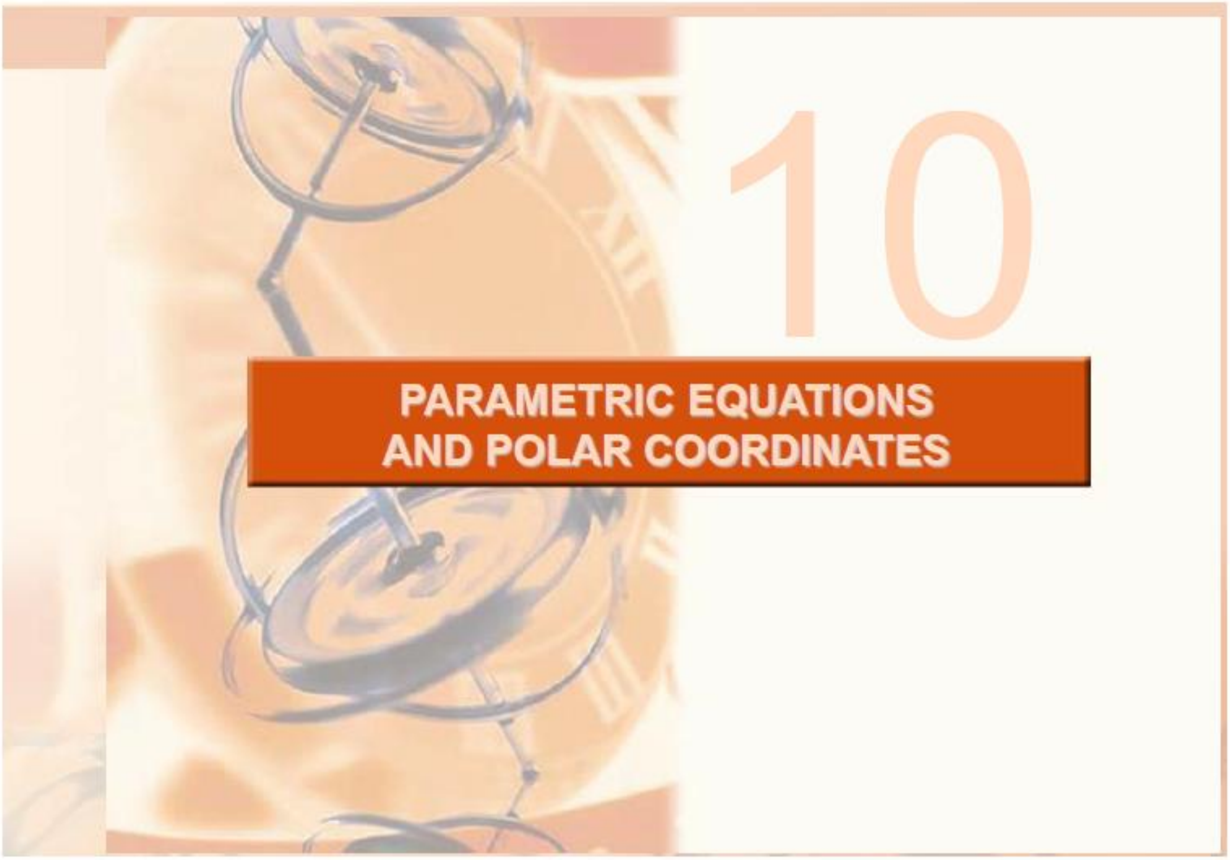


PARAMETRIC EQUATIONS & POLAR COORDINATES

So far, we have described plane curves by giving:

- *y* as a function of *x* [$y = f(x)$] or *x* as a function of *y* [$x = g(y)$]
- A relation between *x* and *y* that defines *y* implicitly as a function of *x* [$f(x, y) = 0$]

PARAMETRIC EQUATIONS & POLAR COORDINATES

In this chapter, we discuss two new methods for describing curves.

PARAMETRIC EQUATIONS

Some curves—such as the cycloid—are best handled when both x and y are given in terms of a third variable t called a parameter [$x = f(t)$, $y = g(t)$].

POLAR COORDINATES

Other curves—such as the cardioid—have their most convenient description when we use a new coordinate system, called the polar coordinate system.

PARAMETRIC EQUATIONS & POLAR COORDINATES

10.1

Curves Defined by Parametric Equations

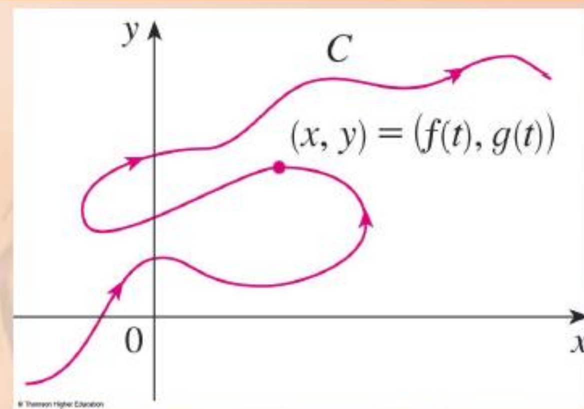
In this section, we will learn about:

Parametric equations and generating their curves.

INTRODUCTION

Imagine that a particle moves along the curve *C* shown here.

- It is impossible to describe *C* by an equation of the form $y = f(x)$.
- This is because *C* fails the Vertical Line Test.



INTRODUCTION

Imagine that a particle moves along the curve *C* shown here.

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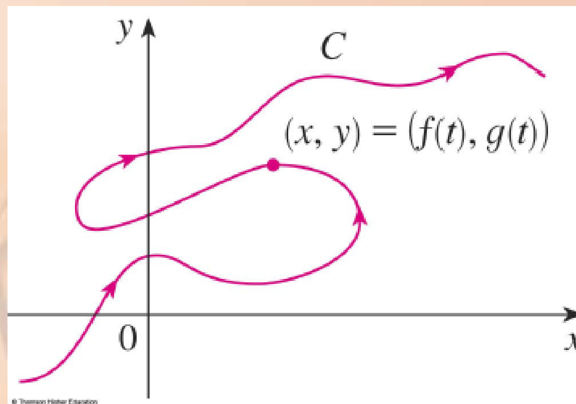
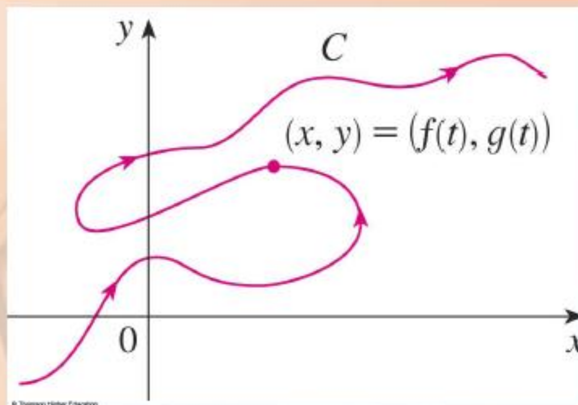


INTRODUCTION

However, the *x*- and *y*-coordinates of the particle are functions of time.

- So, we can write $x = f(t)$ and $y = g(t)$.



INTRODUCTION

Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

PARAMETRIC EQUATIONS

Suppose x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \text{ and } y = g(t)$$

- These are called parametric equations.

PARAMETRIC CURVE

Each value of t determines a point (x, y), which we can plot in a coordinate plane.

As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C.

- This is called a parametric curve.

PARAMETER t

The parameter t does not necessarily represent time.

- In fact, we could use a letter other than t for the parameter.

PARAMETER t

However, in many applications of parametric curves, t does denote time.

- Thus, we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

PARAMETRIC CURVES **Example 1**

Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

PARAMETRIC CURVES — Example 1

Each value of *t* gives a point on the curve, as in the table.

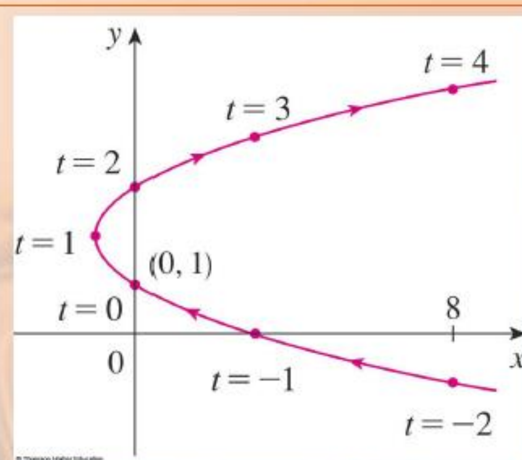
- For instance, if $t = 0$, then $x = 0$, $y = 1$.
- So, the corresponding point is (0, 1).

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5

PARAMETRIC CURVES — Example 1

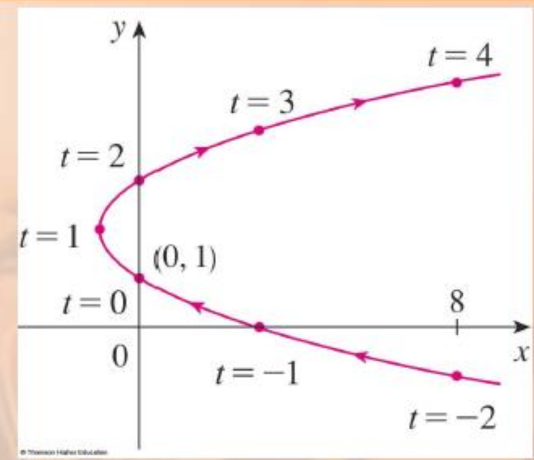
Now, we plot the points (x, y) determined by several values of the parameter, and join them to produce a curve.

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5



PARAMETRIC CURVES **Example 1**

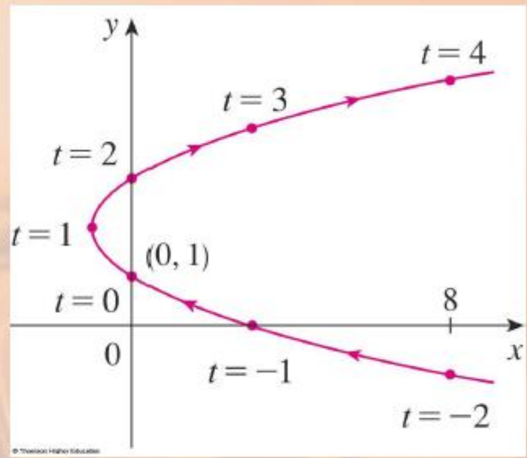
A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases.



PARAMETRIC CURVES **Example 1**

Notice that the consecutive points marked on the curve appear at equal time intervals, but not at equal distances.

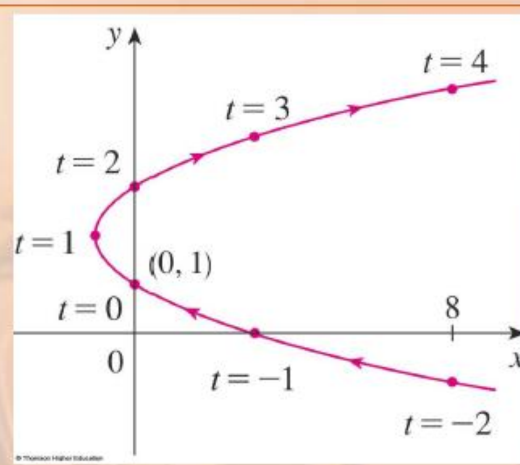
- That is because the particle slows down and then speeds up as *t* increases.



PARAMETRIC CURVES **Example 1**

It appears that the curve traced out by the particle may be a parabola.

- We can confirm this by eliminating the parameter *t*, as follows.



PARAMETRIC CURVES **Example 1**

We obtain $t = y - 1$ from the equation $y = t + 1$.

We then substitute it in the equation $x = t^2 - 2t$.

- This gives: $x = t^2 - 2t$
  $= (y - 1)^2 - 2(y - 1)$
  $= y^2 - 4y + 3$

- So, the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$

PARAMETRIC CURVES

This equation in x and y describes where the particle has been.

- However, it doesn't tell us when the particle was at a particular point.

ADVANTAGES

The parametric equations have an advantage—they tell us when the particle was at a point.

They also indicate the direction of the motion.

PARAMETRIC CURVES

No restriction was placed on the parameter t in Example 1.

So, we assumed t could be any real number.

- Sometimes, however, we restrict t to lie in a finite interval.

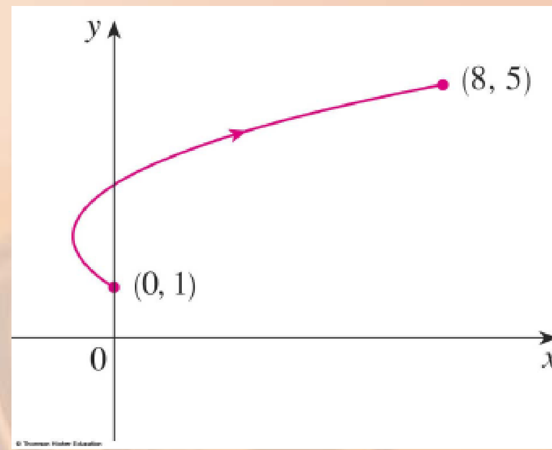
PARAMETRIC CURVES

For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

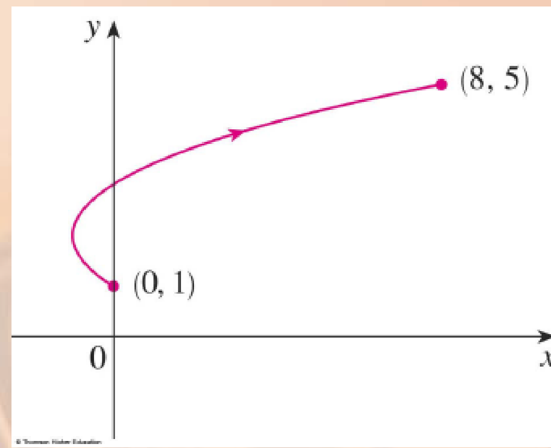
shown is a part of the parabola in Example 1.

- It starts at the point (0, 1) and ends at the point (8, 5).



PARAMETRIC CURVES

The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.



INITIAL & TERMINAL POINTS

In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

PARAMETRIC CURVES Example 2

What curve is represented by the following parametric equations?

$$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$$

PARAMETRIC CURVES Example 2

If we plot points, it appears the curve is a circle.

- We can confirm this by eliminating *t*.

PARAMETRIC CURVES Example 2

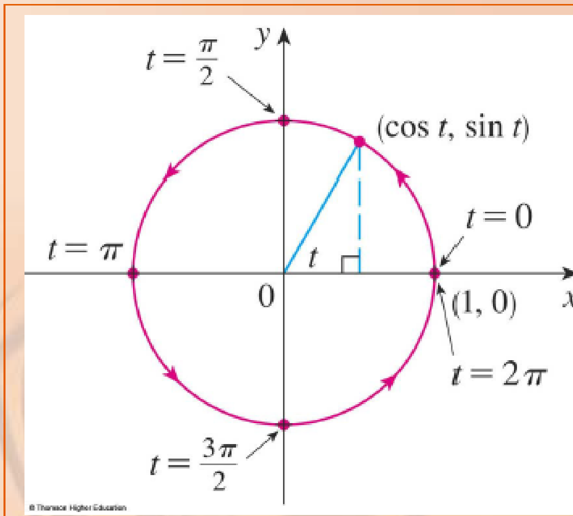
Observe that:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

- Thus, the point (x, y) moves on the unit circle $x^2 + y^2 = 1$

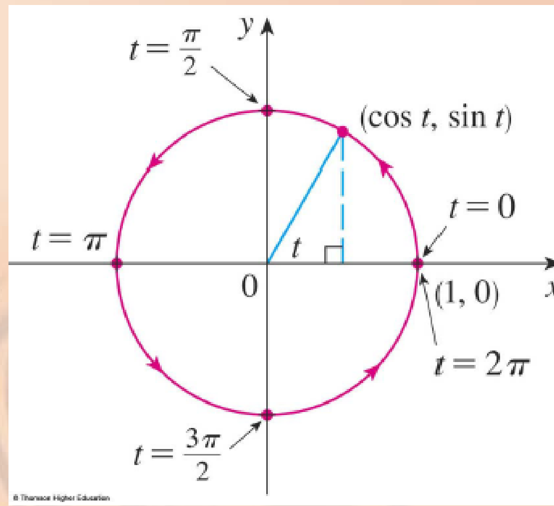
PARAMETRIC CURVES Example 2

Notice that, in this example, the parameter t can be interpreted as the angle (in radians), as shown.



PARAMETRIC CURVES **Example 2**

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point (1, 0).



PARAMETRIC CURVES **Example 3**

What curve is represented by the given parametric equations?

$$x = \sin 2t \quad y = \cos 2t \quad 0 \le t \le 2\pi$$

PARAMETRIC CURVES **Example 3**

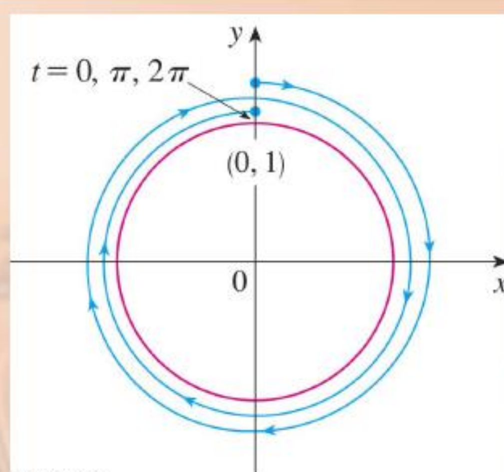
Again, we have:

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

- So, the parametric equations again represent the unit circle $x^2 + y^2 = 1$

PARAMETRIC CURVES **Example 3**

However, as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at (0, 1), moving twice around the circle in the clockwise direction.



PARAMETRIC CURVES

Examples 2 and 3 show that different sets of parametric equations can represent the same curve.

So, we distinguish between:

- A curve, which is a set of points
- A parametric curve, where the points are traced in a particular way

PARAMETRIC CURVES — Example 4

Find parametric equations for the circle with center (h, k) and radius r.

PARAMETRIC CURVES **Example 4**

We take the equations of the unit circle in Example 2 and multiply the expressions for *x* and *y* by *r.*

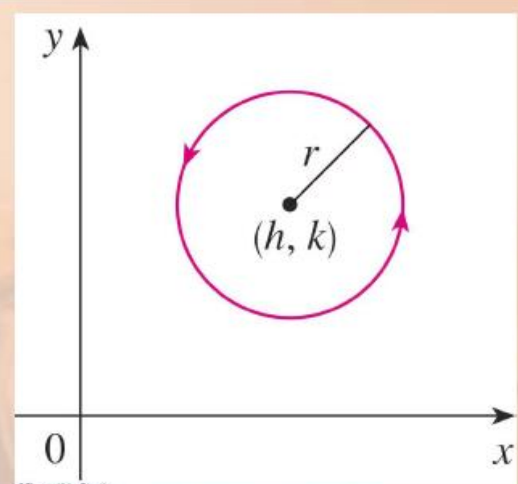
We get:

$$x = r \cos t \qquad y = r \sin t$$

- You can verify these equations represent a circle with radius *r* and center the origin traced counterclockwise.

PARAMETRIC CURVES **Example 4**

Now, we shift *h* units in the *x*-direction and *k* units in the *y*-direction.



PARAMETRIC CURVES — Example 4

Thus, we obtain the parametric equations of the circle with center (h, k) and radius r :

$$x = h + r\cos t \quad y = k + r\sin t \quad 0 \le t \le 2\pi$$

PARAMETRIC CURVES — Example 5

Sketch the curve with parametric equations

$$x = \sin t \qquad y = \sin^2 t$$

Observe that $y = (\sin t)^2 = x^2$.

Thus, the point (x, y) moves on the parabola $y = x2$.

PARAMETRIC CURVES **Example 5**

However, note also that, as $-1 \leq \sin t \leq 1$, we have $-1 \leq x \leq 1$.

- So, the parametric equations represent only the part of the parabola for which $-1 \leq x \leq 1$.

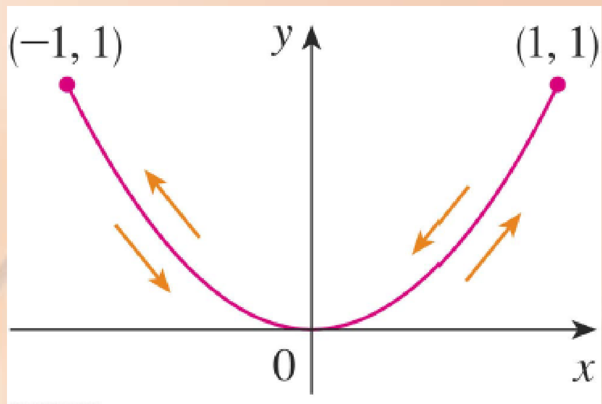
PARAMETRIC CURVES **Example 5**

However, note also that, as $-1 \leq \sin t \leq 1$, we have $-1 \leq x \leq 1$.

- So, the parametric equations represent only the part of the parabola for which $-1 \leq x \leq 1$.

PARAMETRIC CURVES **Example 5**

Since sin *t* is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from (-1, 1) to (1, 1).



GRAPHING DEVICES

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations.

- In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator.
- The points are plotted in order as the corresponding parameter values increase.

GRAPHING DEVICES — Example 6

Use a graphing device to graph the curve

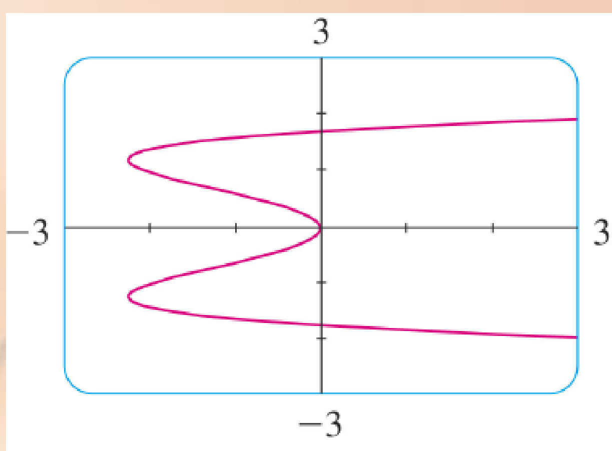
$$x = y^4 - 3y^2$$

- If we let the parameter be $t = y$, we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

GRAPHING DEVICES — Example 6

- Using those parametric equations, we obtain this curve.



GRAPHING DEVICES

Example 6

It would be possible to solve the given equation for y as four functions of x and graph them individually.

- However, the parametric equations provide a much easier method.

GRAPHING DEVICES

In general, if we need to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

GRAPHING DEVICES

Notice also that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

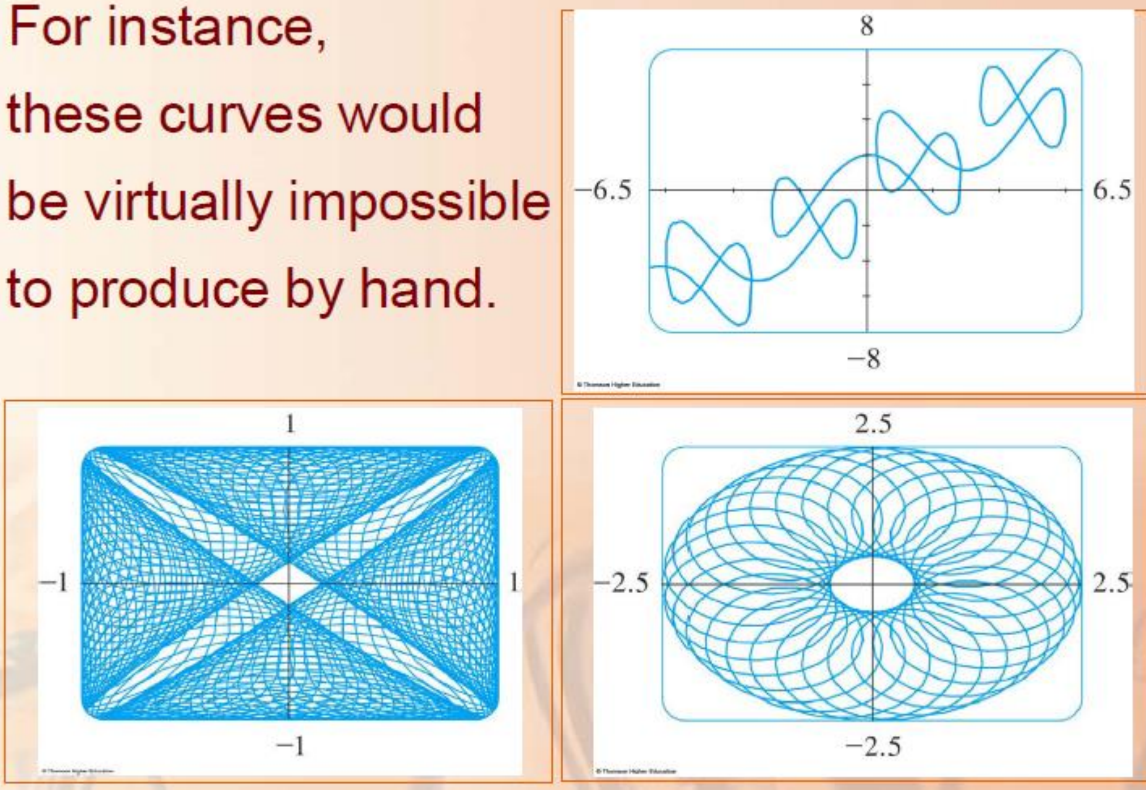
$$x = t \qquad y = f(t)$$

GRAPHING DEVICES

Graphing devices are particularly useful when sketching complicated curves.

COMPLEX CURVES

For instance, these curves would be virtually impossible to produce by hand.



CAD

One of the most important uses of parametric curves is in computer-aided design (CAD).

BÉZIER CURVES

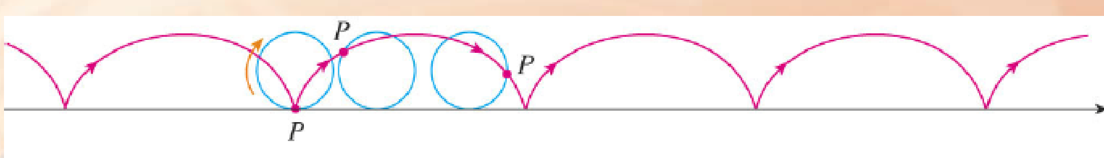
In the Laboratory Project after Section 10.2, we will investigate special parametric curves called Bézier curves.

- These are used extensively in manufacturing, especially in the automotive industry.
- They are also employed in specifying the shapes of letters and other symbols in laser printers.

CYCLOID

Example 7

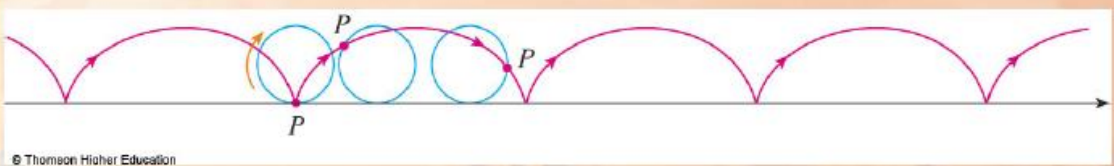
The curve traced out by a point *P* on the circumference of a circle as the circle rolls along a straight line is called a cycloid.



CYCLOIDS **Example 7**

Find parametric equations for the cycloid if:

- The circle has radius *r* and rolls along the *x*-axis.
- One position of *P* is the origin.



CYCLOIDS **Example 7**

We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when *P* is at the origin).

Suppose the circle has rotated through θ radians.

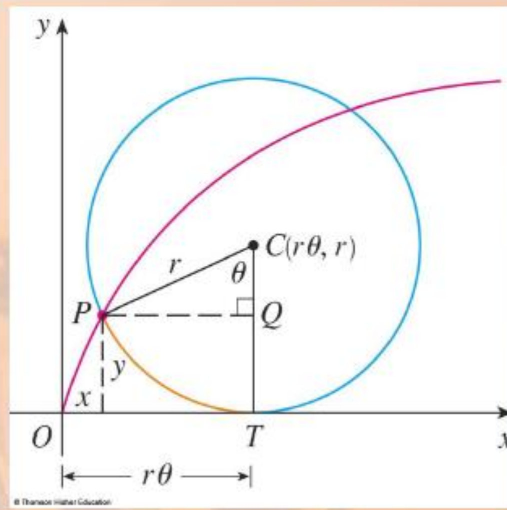
CYCLOIDS

Example 7

As the circle has been in contact with the line, the distance it has rolled from the origin is:

$|OT| = \text{arc } PT = r\theta$

- Thus, the center of the circle is $C(r\theta, r)$.



CYCLOIDS

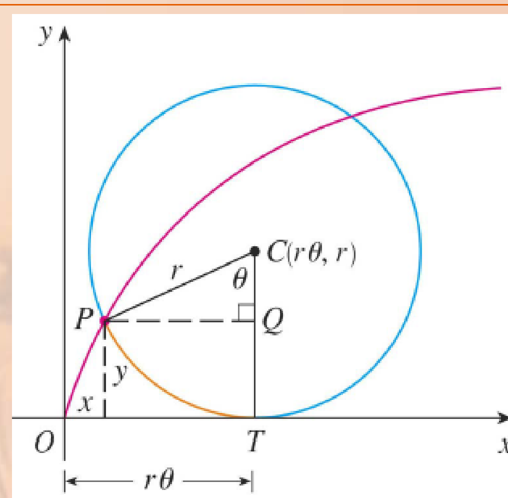
Example 7

Let the coordinates of P be (x, y).

Then, from the figure, we see that:

- $x = |OT| - |PQ|$
  $= r\theta - r\sin\theta$
  $= r(\theta - \sin\theta)$

- $y = |TC| - |QC|$
  $= r - r\cos\theta$
  $= r(1 - \cos\theta)$



CYCLOIDS **E. g. 7—Equation 1**

Therefore, parametric equations of the cycloid are:

$$x = r(\theta - \sin \theta) \qquad y = r(1 - \cos \theta) \quad \theta \in \mathbb{R}$$

CYCLOIDS **Example 7**

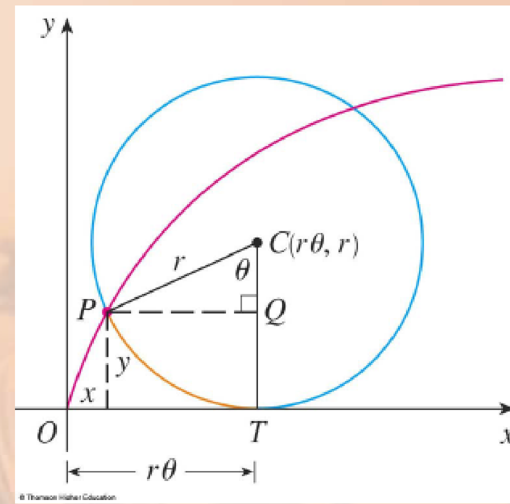
One arch of the cycloid comes from one rotation of the circle.

So, it is described by $0 \leq \theta \leq 2\pi$.

CYCLOIDS — Example 7

Equations 1 were derived from the figure, which illustrates the case where $0 < \theta < \pi/2$.

- However, it can be seen that the equations are still valid for other values of θ.



PARAMETRIC VS. CARTESIAN — Example 7

It is possible to eliminate the parameter θ from Equations 1.

However, the resulting Cartesian equation in x and y is:

- Very complicated
- Not as convenient to work with

CYCLOIDS

One of the first people to study the cycloid was Galileo.

- He proposed that bridges be built in the shape.
- He tried to find the area under one arch of a cycloid.

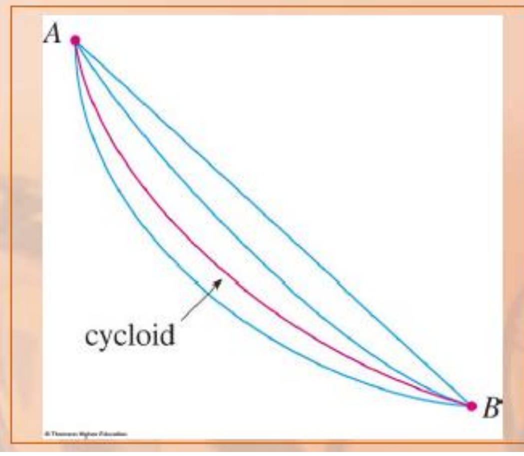
BRACHISTOCHRONE PROBLEM

Later, this curve arose in connection with the brachistochrone problem—proposed by the Swiss mathematician John Bernoulli in 1696:

- Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point *A* to a lower point *B* not directly beneath *A*.

BRACHISTOCHRONE PROBLEM

Bernoulli showed that, among all possible curves that join *A* to *B*, the particle will take the least time sliding from *A* to *B* if the curve is part of an inverted arch of a cycloid.



TAUTOCHRONE PROBLEM

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the tautochrone problem:

- No matter where a particle is placed on an inverted cycloid, it takes the same time to slide to the bottom.

CYCLOIDS & PENDULUMS

He proposed that pendulum clocks (which he invented) swing in cycloidal arcs.

- Then, the pendulum takes the same time to make a complete oscillation—whether it swings through a wide or a small arc.

PARAMETRIC CURVE FAMILIES **Example 8**

Investigate the family of curves with parametric equations

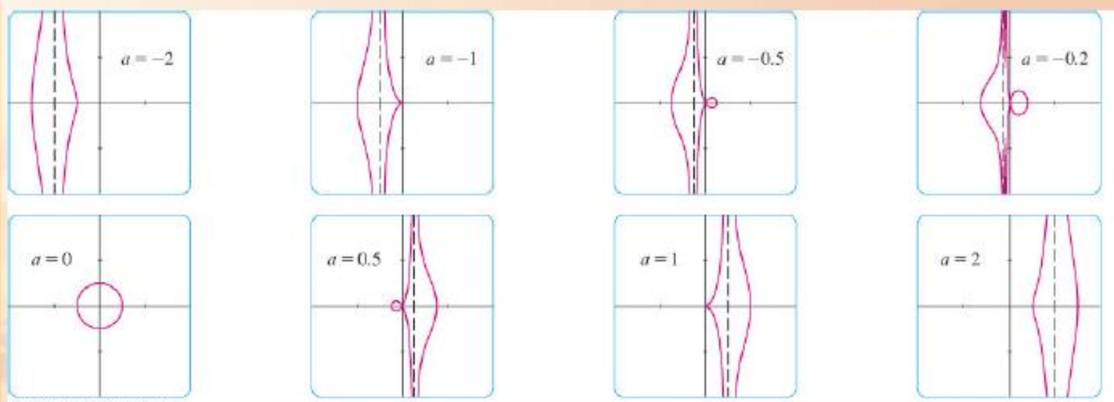
$$x = a + \cos t \qquad y = a \tan t + \sin t$$

- What do these curves have in common?
- How does the shape change as *a* increases?

PARAMETRIC CURVE FAMILIES **Example 8**

We use a graphing device to produce the graphs for the cases *a* =

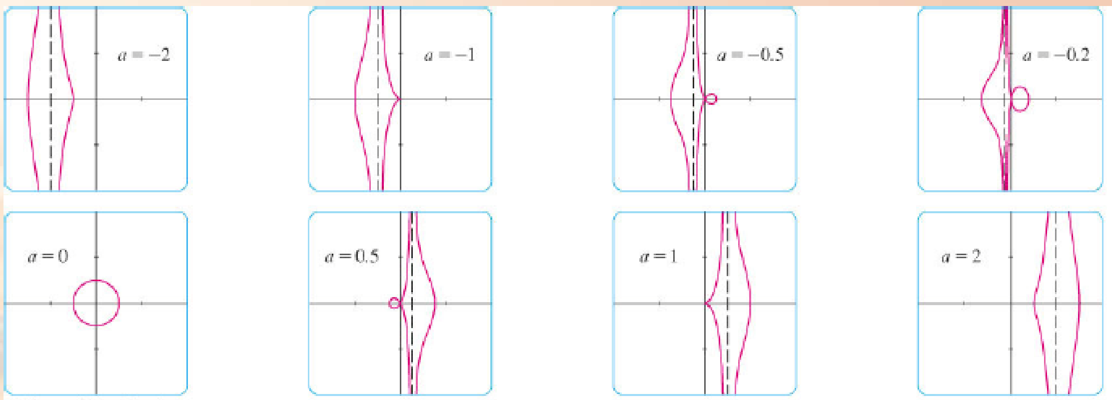
-2, -1, -0.5, -0.2, 0, 0.5, 1, 2



PARAMETRIC CURVE FAMILIES **Example 8**

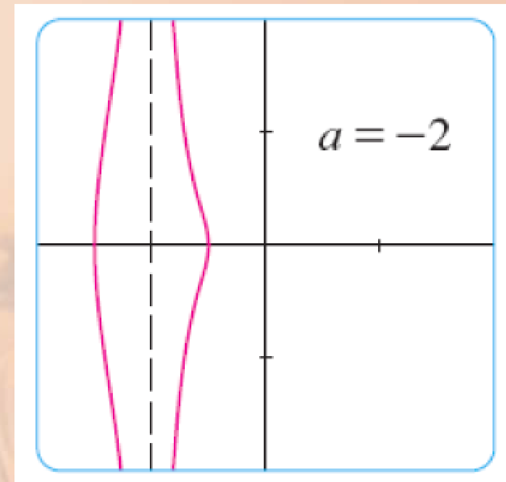
Notice that:

- All the curves (except for $a = 0$) have two branches.
- Both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.



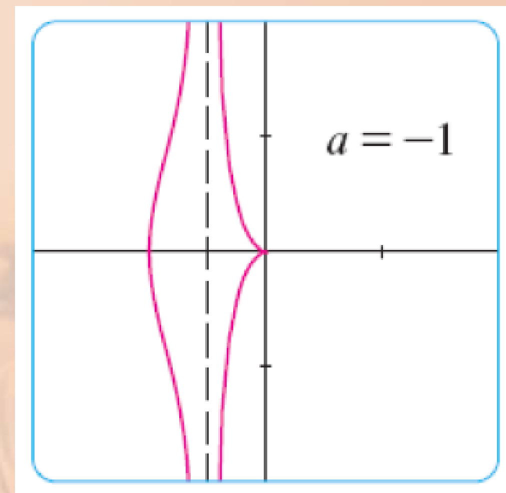
LESS THAN -1 **Example 8**

When *a* < -1, both branches are smooth.



REACHES -1 **Example 8**

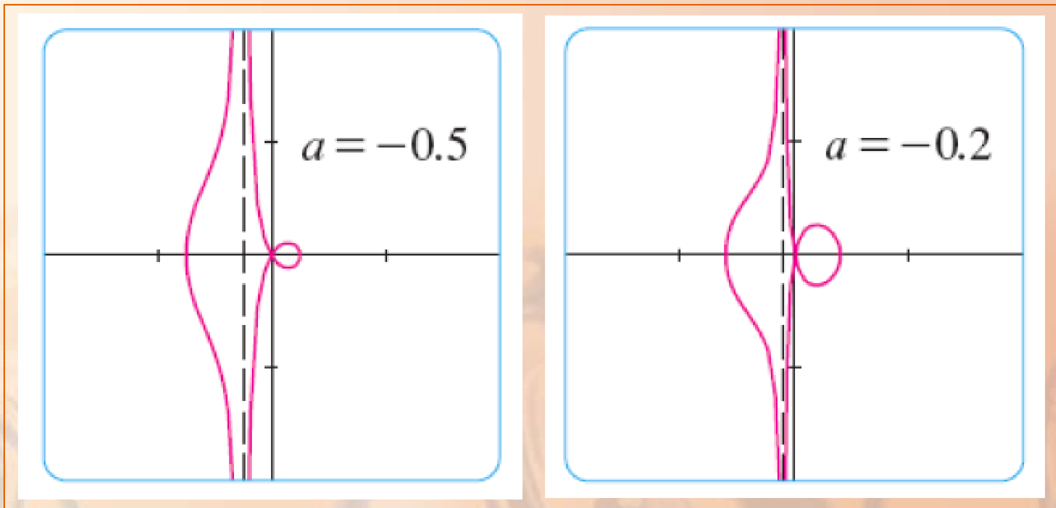
However, when *a* reaches -1, the right branch acquires a sharp point, called a cusp.



BETWEEN -1 AND 0

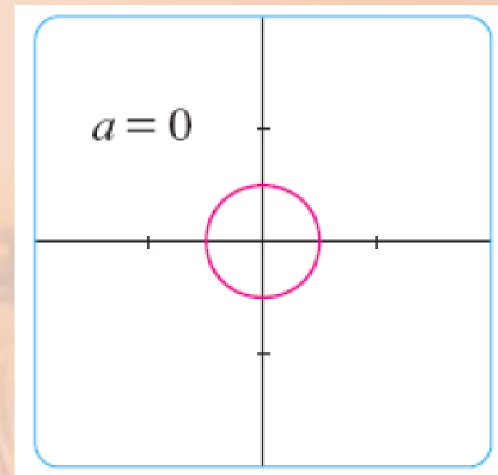
Example 8

For *a* between -1 and 0, the cusp turns into a loop, which becomes larger as *a* approaches 0.



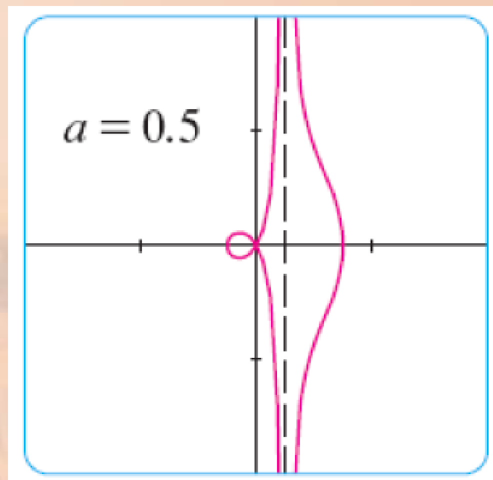
EQUALS 0 Example 8

When $a = 0$, both branches come together and form a circle.

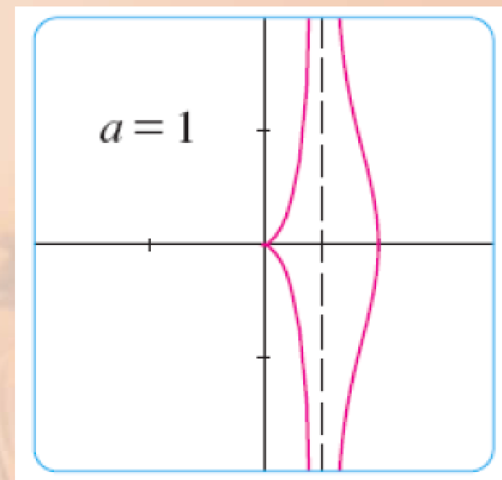


BETWEEN 0 AND 1 **Example 8**

For *a* between 0 and 1, the left branch has a loop.



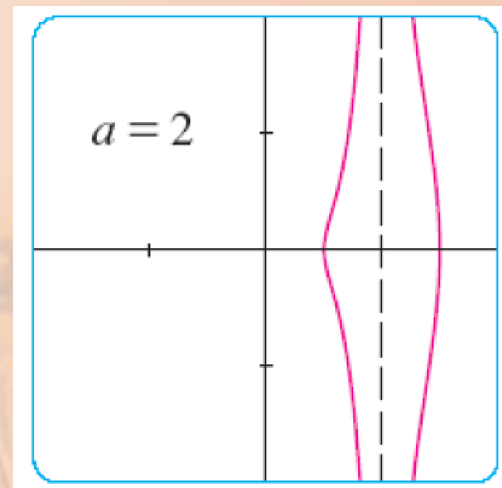
When a = 1, the loop shrinks to become a cusp.



GREATER THAN 1 — Example 8

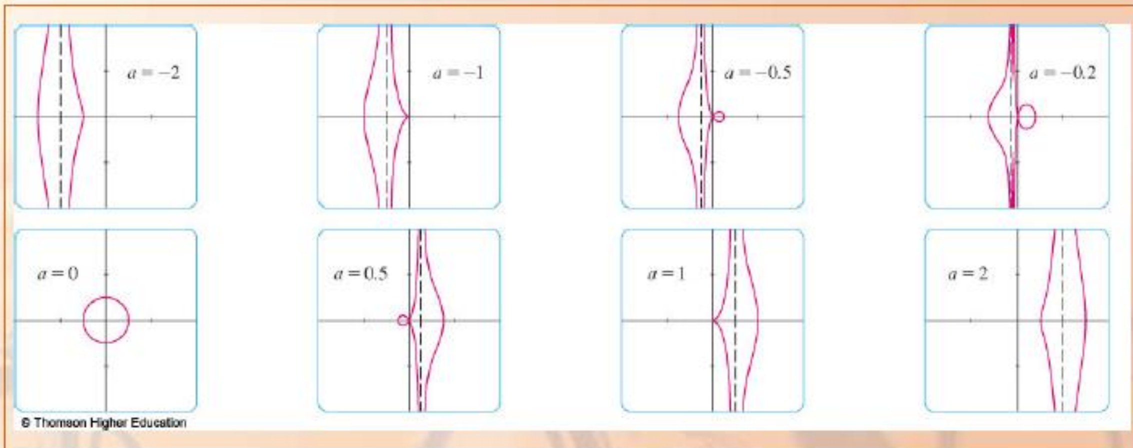
For *a* > 1, the branches become smooth again.

As *a* increases further, they become less curved.



PARAMETRIC CURVE FAMILIES Example 8

Notice that curves with *a* positive are reflections about the *y*-axis of the corresponding curves with *a* negative.



CONCHOIDS OF NICOMEDES Example 8

These curves are called conchoids of Nicomedes—after the ancient Greek scholar Nicomedes.

CONCHOIDS OF NICOMEDES **Example 8**

He called them so because the shape of their outer branches resembles that of a conch shell or mussel shell.