10.2 Parametric Equations

Let x and y be two continuous functions of a third variable t defined on an interval I. For each value of t in I, we can plot a point (x, y) in the Cartesian coordinates system. The collection of all these points is a plane curve which we call a **plane curve** or a **parametric curve**. The equations

$$x = f(t) \qquad y = g(t), \quad t \text{ in } I$$

are called **parametric equations** and the variable t is called a **parameter**. Each point on the curve is determined from a chosen value of t. Plotting all these points in increasing value of t, the plane curve is traced in a certain direction, called its **orientation**. That is, a plane curve is an oriented curve.

Example 10.2.1
Sketch the parametric curve with parametric equations:

$$x = \cos t \qquad y = \sin t, \quad 0 \leq t \leq 2\pi.$$

Solution.
By eliminating the parameter t we see that the point (x, y) is on the unit circle:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

As shown in Figure 10.2.1, the curve starts with the initial point $(1, 0)$ and goes along the unit circle counterclockwise and it ends at the terminal point $(1, 0)$ ■

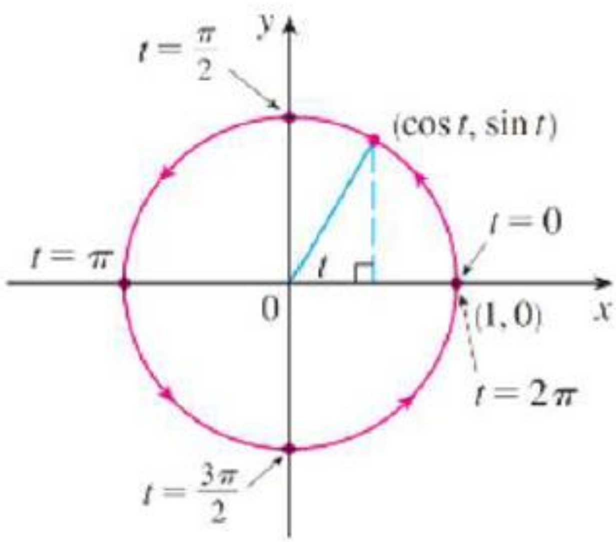

Figure 10.2.1

Example 10.2.2
Sketch the parametric curve with parametric equations:

$$x = \sin 2t \qquad y = \cos 2t, \quad 0 \leq t \leq 2\pi.$$

Solution.
By eliminating the parameter t we see that the point (x, y) is on the unit circle:

$$x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1.$$

As shown in Figure 10.2.2, the curve starts with the initial point $(0, 1)$ and goes along the unit circle clockwise and traces the circle twice and it ends at the terminal point $(0, 1)$ ■

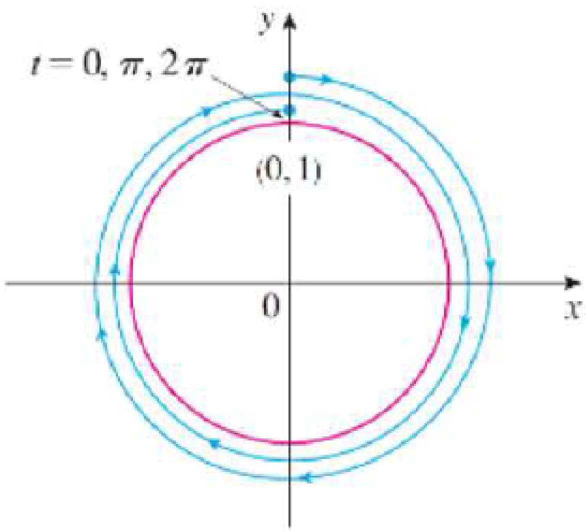

Figure 10.2.2

Remark 10.2.1
The two examples above show that different sets of parametric equations can represent the same curve. Thus, we distinguish between a *curve*, which is a set of points in the Cartesian system, and a *parametric curve* in which the points are traced in a particular way. For example, different parametric equations can be used to represent various speeds at which objects travel along a path.

The above two examples shows how to find the rectangular equation of a plane curve given the parametric equations. This method is known as the method of **eliminating the parameter**. It is important to remember that the range of x and y in the rectangular equation may be altered by this change as illustrated in the next example.

Example 10.2.3
Sketch the curve

$$x = \frac{1}{\sqrt{t+1}} \qquad y = \frac{t}{t+1},\ t > -1.$$

Solution.
Solving for t in the first equation, we find

$$\begin{aligned} x =& \frac{1}{\sqrt{t+1}} \\ x^2 =& \frac{1}{t+1} \\ t+1 =& \frac{1}{x^2} \\ t =& \frac{1-x^2}{x^2}. \end{aligned}$$

Plugging this expression of t in the second equation, we find

$$y = \frac{\frac{1-x^2}{x^2}}{\frac{1-x^2}{x^2}+1} = 1 - x^2.$$

which is the parabola shown in Figure 10.2.3.

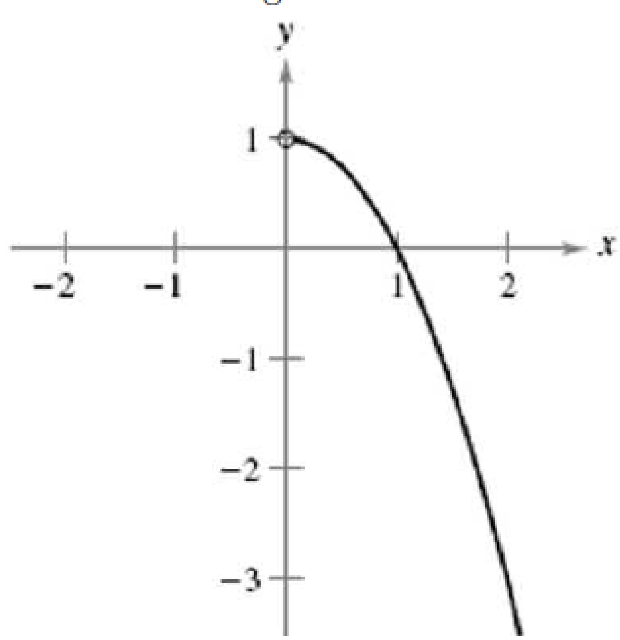

Figure 10.2.3

Notice the the rectangular equation $y = 1 - x^2$ is defined for all values of x. However, the condition $t > -1$ implies that $x > 0$ ■

Example 10.2.4
Find the parametric equations of the circle centered at (h, k) and with radius r and traced counterclockwise once.

Solution.
In the Cartesian coordinates system, the unit circle is given by

$$(x-h)^2 + (y-k)^2 = r^2.$$

Thus, the parametric equations are given by

$$x - h = r\cos t \Longrightarrow x = h + r\cos t$$

and

$$y - k = r\sin t \Longrightarrow y = k + r\sin t$$

where $0 \le t \le 2\pi$ ■

Example 10.2.5
Find the parametric equations of each of the following curves:
(a) $x = y^2 - 2y + 5$
(b) $y = x^2 - 4x + 3$.

Solution.
(a) One parametric representation is

$$x = t^2 - 2t + 5 \quad \text{and} \quad y = t.$$

(b) A parametric representation is

$$x = t \quad \text{and} \quad y = t^2 - 4t + 3 \blacksquare$$

Example 10.2.6
Determine the curve traced by a point P on the circumference of a circle of radius r rolling along a straight line in a plane. Such a curve is called a **cycloid.**

Solution.

Let the parameter be the angle of rotation θ for our given circle. Note that $\theta = 0$ when the point P is at the origin. Next consider the distance the circle has rolled from the origin after it has rotated through θ radians, which is

given by $|OT| = |\text{arc}PT| = r\theta$. See Figure 10.2.4.

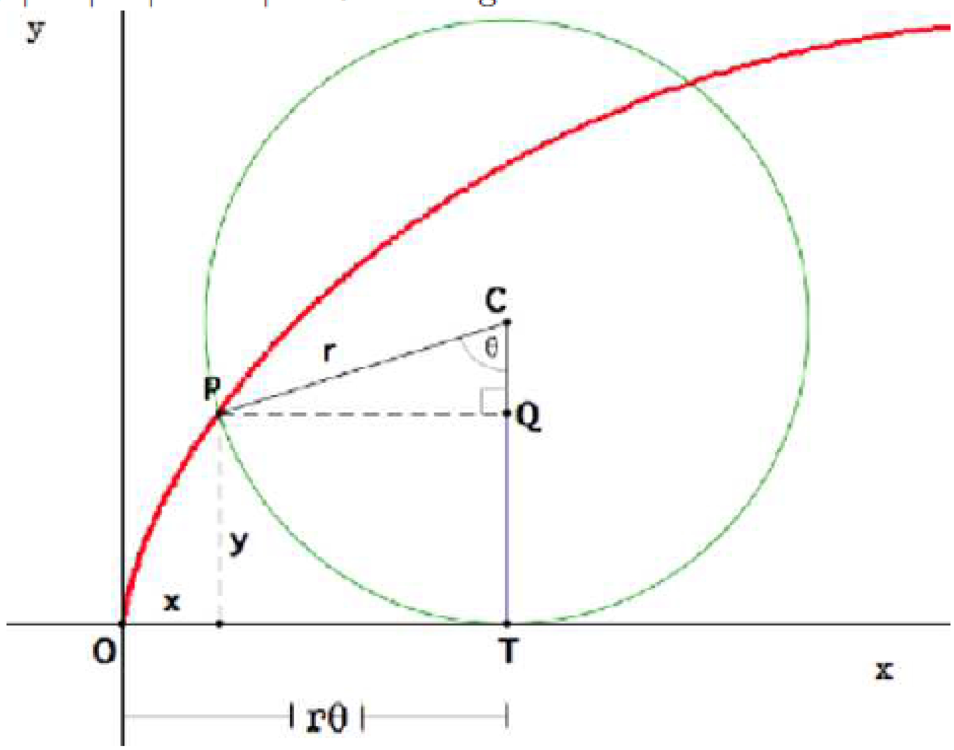

Figure 10.2.4

The coordinates of the center are $C(r\theta, r)$. Letting x and y be the coordinates of P, we find

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(1 - \sin\theta)$$

and

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

which give us the parametric equations of the cycloid ■

Graphing Parametric Curves with TI

1. Press Mode key
2. Select Par and hits Enter
3. Press $Y =$ and enter the equations
4. Hit Graph