Calculus with Parametric equations

Let $\mathcal{C}$ be a parametric curve described by the parametric equations $x = f(t), y = g(t)$. If the function f and g are differentiable and y is also a differentiable function of x, the three derivatives $\frac{dy}{dx}$, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related by the Chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

using this we can obtain the formula to compute $\frac{dy}{dx}$ from $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

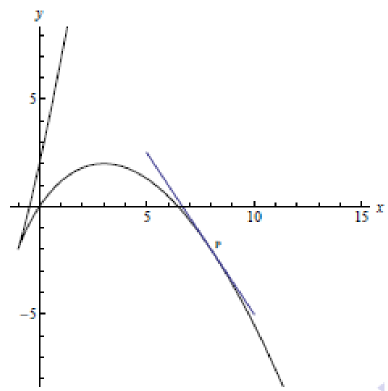
- The value of $\frac{dy}{dx}$ gives gives the slope of a tangent to the curve at any given point. This sometimes helps us to draw the graph of the curve.
- The curve has a **horizontal tangent** when $\frac{dy}{dx} = 0$, and has a **vertical tangent** when $\frac{dy}{dx} = \infty$.
- The second derivative $\frac{d^2y}{dx^2}$ can also be obtained from $\frac{dy}{dx}$ and $\frac{dx}{dt}$. Indeed,

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Example 1

Example 1 (a) Find an equation of the tangent to the curve
$x = t^2 - 2t \qquad y = t^3 - 3t \qquad \text{when} \qquad t = -2$

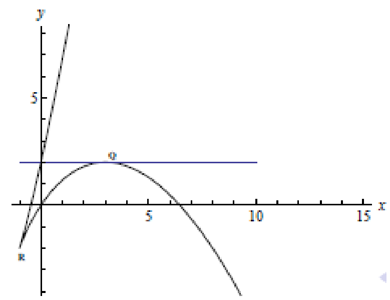
- When $t = -2$, the corresponding point on the curve is $P = (4 + 4, -8 + 6) = (8, -2)$.
- We have $\frac{dx}{dt} = 2t - 2$ and $\frac{dy}{dt} = 3t^2 - 3$.
- Therefore $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t-2}$ when $2t - 2 \neq 0$.
- When $t = -2$, $\frac{dy}{dx} = \frac{12-3}{-4-2} = \frac{9}{-6} = -\frac{3}{2}$.
- The equation of the tangent line at the point P is $(y + 2) = -\frac{3}{2}(x - 8)$.



Example 1

Example 1 (b) Find the point on the parametric curve where the tangent is horizontal $\quad x = t^2 - 2t \qquad y = t^3 - 3t$

- From above, we have that $\frac{dy}{dx} = \frac{3t^2-3}{2t-2}$.
- $\frac{dy}{dx} = 0$ if $\frac{3t^2-3}{2t-2} = 0$ if $3t^2 - 3 = 0$ (and $2t - 2 \neq 0$).
- Now $3t^2 - 3 = 0$ if $t = \pm 1$.
- When $t = -1$, $2t - 2 \neq 0$ and therefore the graph has a horizontal tangent. The corresponding point on the curve is $Q = (3, 2)$.
- When $t = 1$, we have $\frac{dx}{dt} = 2t - 2 = 0$ and there is not a well defined tangent. If the curve describes the motion of a particle, this is a point where the particle has stooped. In this case, we see that the corresponding point on the curve is $R = (-1, -2)$ and the curve has a cusp(sharp point).



Example 1

Example 1 (c) Does the parametric curve given below have a vertical tangent?
$x = t^2 - 2t \qquad y = t^3 - 3t$

- From above, we have that $\frac{dy}{dx} = \frac{3t^2-3}{2t-2}$.
- The curve has a vertical tangent if $2t - 2 = 0$ (and $3t^2 - 3 \neq 0$).
- $dx/dt = 2t - 2 = 0$ if $t = 1$, however in this case $dy/dt = 3t^2 - 3 = 0$, hence the curve does not have a vertical tangent.

(d) Use the second derivative to determine where the graph is concave up and concave down.

- $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ if $\frac{dx}{dt} \neq 0$
- If $\frac{dx}{dt} \neq 0$, we have $\frac{dy}{dx} = \frac{3t^2-3}{2t-2} = \frac{3}{2}(t+1)$.
- Therefore $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{3}{2}(t+1))}{2t-2} = \frac{3}{4(t-1)}$
- We see that $\frac{d^2y}{dx^2} > 0$ if $t > 1$ and $\frac{d^2y}{dx^2} < 0$ if $t < 1$.
- Therefore the graph is concave up if $t < 1$ and concave down if $t > 1$. (when $t = 1$, the point on the curve is at the cusp).

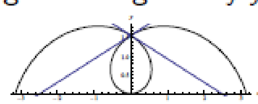
Example 2

Consider the curve $\mathcal{C}$ defined by the parametric equations

$$x = t\cos t \qquad y = t\sin t \qquad -\pi \leq t \leq \pi$$

Find the equations of both tangents to $\mathcal{C}$ at $(0, \frac{\pi}{2})$

- We first find the value(s) of t which correspond to this point. At this point, $t\cos t = 0$, therefore, either $t = 0$ or $\cos t = 0$and $t = \pm\frac{\pi}{2}$. When $t = 0$, the corresponding point on the curve is $(0, 0)$ and when $t = \pm\frac{\pi}{2}$, the corresponding point is $(0, \frac{\pi}{2})$.
- We have $\frac{dy}{dt} = \sin t + t\cos t$ and $\frac{dx}{dt} = \cos t - t\sin t$.
- Therefore $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t - t\cos t}{\cos t + t\sin t}$.
- When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1-0}{0-\frac{\pi}{2}} = \frac{-2}{\pi}$
- When $t = \frac{-\pi}{2}$, $\frac{dy}{dx} = \frac{-1-0}{0-(-\frac{\pi}{2})(-1)} = \frac{2}{\pi}$
- The equations of the tangents are given by $y - \frac{\pi}{2} = \frac{-2}{\pi}x$ and $y - \frac{\pi}{2} = \frac{2}{\pi}x$.



Area under a curve

Recall that the area under the curve $y = F(x)$ where $a \leq x \leq b$ and $F(x) > 0$ is given by

$$\int_a^b F(x)dx$$

If **this curve (of form** $y = F(x), F(x) > 0, \quad a \leq x \leq b$**)** can be traced out **once** by parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$ then we can calculate the area under the curve by computing the integral:

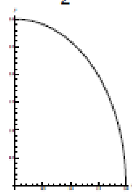
$$\boxed{\left| \int_\alpha^\beta g(t)f'(t)dt \right| = \int_\alpha^\beta g(t)f'(t)dt \qquad \text{or} \qquad \int_\beta^\alpha g(t)f'(t)dt}$$

Area under a curve

Example Find the area under the curve

$$x = 2\cos t \qquad y = 3\sin t \qquad 0 \le t \le \frac{\pi}{2}$$

- The graph of this curve is a quarter ellipse, starting at $(2, 0)$ and moving counterclockwise to the point $(0, 3)$.
- From the formula, we get that the area under the curve is $\left|\int_\alpha^\beta g(t)f'(t)dt\right|$.
- $\int_\alpha^\beta g(t)f'(t)dt = \int_0^{\pi/2} 3\sin t(2(-\sin t))dt$
  $= -6\int_0^{\pi/2} \sin^2 t dt = -6\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos(2t))dt$
  $= -3[t - \frac{\sin(2t)}{2}]_0^{\frac{\pi}{2}} = -3[\frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + \frac{\sin 0}{2}] = -3[\frac{\pi}{2} - 0] = \frac{-3\pi}{2} = -\frac{3\pi}{2}$.
- Therefore the area under the curve is $\frac{3\pi}{2}$.



Arc Length: Length of a curve

If a curve $\mathcal{C}$ is given by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where the derivatives of f and g are continuous in the interval $\alpha \leq t \leq \beta$ and $\mathcal{C}$ is traversed exactly once as t increases from α to β, then we can compute the length of the curve with the following integral:

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{\alpha}^{\beta} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} dt}$$

- If the curve is of the form $y = F(x)$, $\ a \leq x \leq b$, this formula can be derived from our previous formula

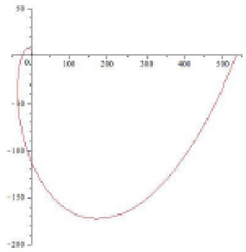
$$L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

using the reverse substitution, $x = f(t)$, giving $\frac{dx}{dt} = f'(t)$.

Example

Example Find the arc length of the spiral defined by

$$x = e^t \cos t \qquad y = e^t \sin t \qquad 0 \le t \le 2\pi$$



- $x'(t) = e^t \cos t - e^t \sin t, \quad y'(t) = e^t \sin t + e^t \cos t.$
- $L = \int_0^{2\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} dt$
- $= \int_0^{2\pi} e^t \sqrt{\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t} dt$
- $= \int_0^{2\pi} e^t \sqrt{2} dt = \sqrt{2} e^t \Big|_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1).$

Example

Example Find the arc length of the circle defined by

$$x = \cos 2t \qquad y = \sin 2t \qquad 0 \le t \le 2\pi$$

Do you see any problems?

- If we apply the formula $L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$, then, we get
- $L = \int_0^{2\pi} \sqrt{4\sin^2 2t + 4\cos^2 2t} dt$
- $= 2\int_0^{2\pi} \sqrt{1} dt = 4\pi$
- The problem is that this parametric curve traces out the circle twice, so we get twice the circumference of the circle as our answer.