Parametric Curves

Mathematics 54–Elementary Analysis 2

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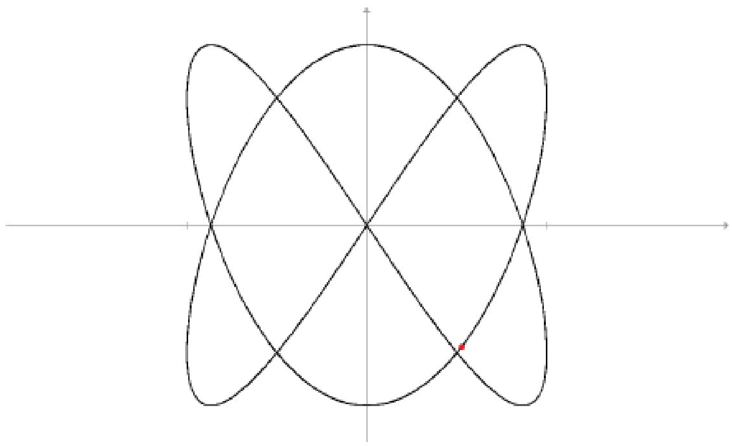
What are parametric equations?

Consider a particle moving along the curve below



What are parametric equations?

The curve cannot be expressed as the graph of an equation of the form $y = f(x)$.

Sometimes, it's impossible to find an equation relating the $x-$ and $y-$coordinates of the particle.

But we can express the $x-$ and the $y-$component of the particle's position as functions of time t, i.e.

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

In this process, we say that we are parametrizing the curve above.

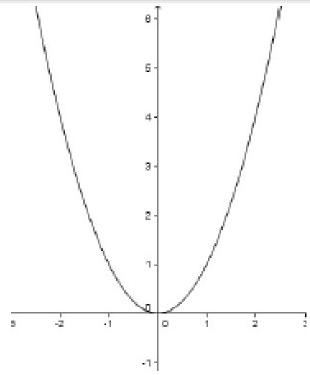
The equations defining x and y as functions of t are called parametric equations, and t is called a parameter.

Parametric Curves

Example

Sketch the curve defined by $C: x = t, \; y = t^2$, where $t \in \mathbb{R}$.

t		(x, y)
-2	$\to$	$(-2, 4)$
-1	$\to$	$(-1, 1)$
0	$\to$	$(0, 0)$
1	$\to$	$(1, 1)$
2	$\to$	$(2, 4)$
3	$\to$	$(3, 9)$



Indeed, if we relate x and y directly by eliminating t, we have $y = x^2$.

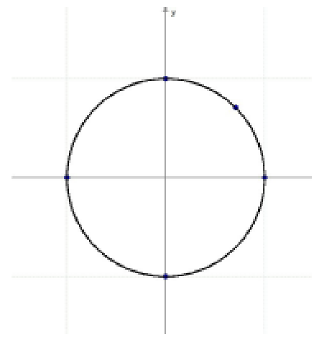
We have just obtained a cartesian equivalent of the parametric curve.

Parametric Curves

Example

Sketch the curve defined by $x = \cos t$ and $y = \sin t$, where $0 \leqslant t \leqslant 2\pi$.

t		(x, y)
0	$\rightarrow$	$(1, 0)$
$\frac{\pi}{4}$	$\rightarrow$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$\frac{\pi}{2}$	$\rightarrow$	$(0, 1)$
π	$\rightarrow$	$(-1, 0)$
$\frac{3\pi}{2}$	$\rightarrow$	$(0, -1)$
2π	$\rightarrow$	$(1, 0)$



Indeed, if we relate x and y directly, eliminating t, we have $x^2 + y^2 = 1$.

The direction of trace is known as the direction of increasing parameter.
In this case, it's counterclockwise.

The point corresponding to $t = 0$ is the *initial point*.
The point corresponding to $t = 2\pi$ is the *terminal point*.

Parametric Curves

Remark

The function $y = f(x)$ can be parametrized as

$$\begin{cases} x &= t \\ y &= f(t) \end{cases}$$

Example

Parametrize the line connecting the points $(0,1)$ and $(3,7)$.

Using the two-point form of a line,

$$y - 1 = \frac{7-1}{3-0}(x - 0) \quad \Leftrightarrow \quad y = 2x + 1$$

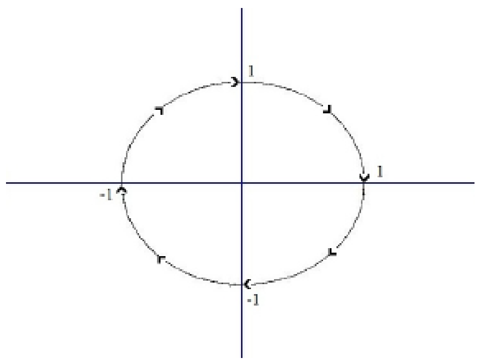
Hence, we can parametrize the line as

$$\begin{cases} x &= t \\ y &= 2t + 1 \end{cases}$$

Examples

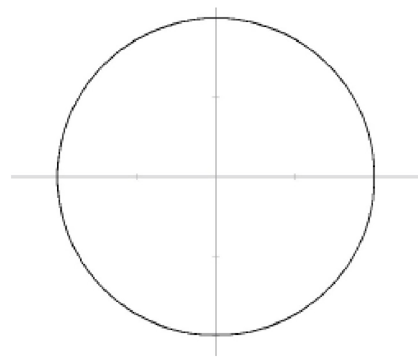
Parametrization of circles...

$$\begin{aligned} x &= \sin t \\ y &= \cos t \\ & 0 \leqslant t \leqslant 2\pi \end{aligned}$$



'clockwise' circle

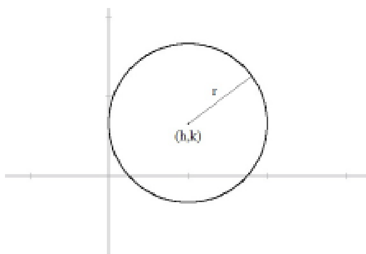
$$\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \\ & 0 \leqslant t \leqslant 2\pi \end{aligned}$$



'counterclockwise' circle of radius 2 units

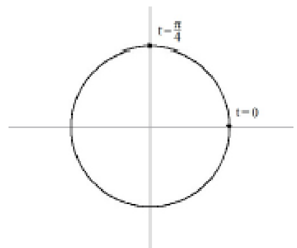
Examples

$$\begin{aligned} x &= h + r\cos t \\ y &= k + r\sin t \\ & \quad 0 \leqslant t \leqslant 2\pi \end{aligned}$$



circle of radius r centered at (h, k)

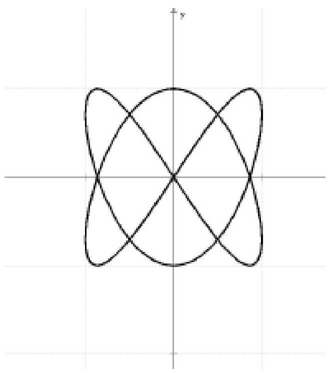
$$\begin{aligned} x &= \cos 2t \\ y &= \sin 2t \\ & \quad 0 \leqslant t \leqslant \pi \end{aligned}$$



'faster' circle...

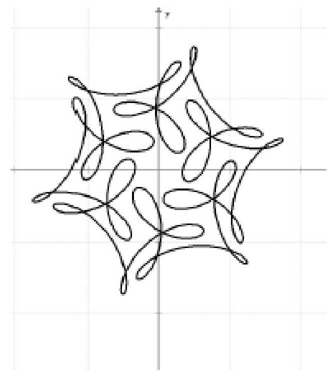
Interesting Examples

$$x = \sin 2t$$
$$y = \sin 3t$$



Lissajous curves

$$x = \cos t + 0.5\cos 7t + \left(\frac{1}{3}\right)\sin 17t$$
$$y = \sin t + 0.5\sin 7t + \left(\frac{1}{3}\right)\cos 17t$$
$$0 \leqslant t \leqslant 2\pi$$



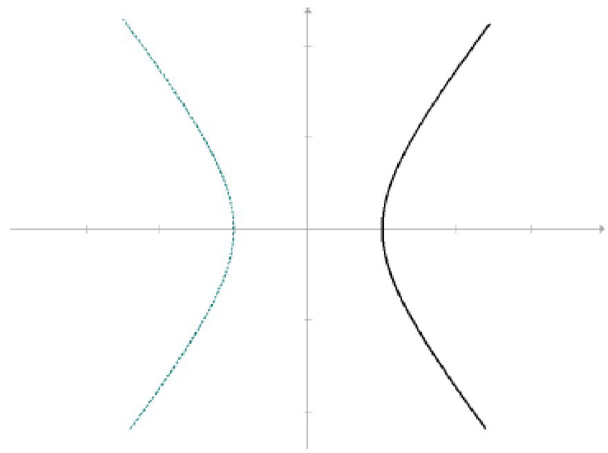
Examples

Consider the curve defined by

$$x = \sec t$$

$$y = \tan t$$

$$-\frac{\pi}{2} < t < \frac{\pi}{2}.$$



Note that

$$x^2 = \sec^2 t$$

$$y^2 = \tan^2 t$$

Thus,

$$x^2 - y^2 = 1$$

Exercises

1. Describe the curve given by the parametric equations $x = 2\cos t$ and $y = 3\sin t$.
2. Given $A(2,0)$ and $B(-1,1)$. Give a parametrization for
   1. the line containg A and B.
   2. the line segment connecting A and B.
3. What can be said about the parametric equations $x = \cos t, y = -\sin t$ and $x = \cos 2t, y = -\sin 2t$?

Slope of Parametric Curves

Smooth Curve

A parametric curve $C: x = f(t), y = g(t)$ is said to be smooth if f and g are differentiable and no value of t satisfies $f'(t) = g'(t) = 0$.

If C is a smooth curve, then its slope $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Also, the concavity $\frac{d^2y}{dx^2}$ is given by

$$\frac{d^2y}{dx^2} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}} \neq \frac{y''(t)}{x''(t)}$$

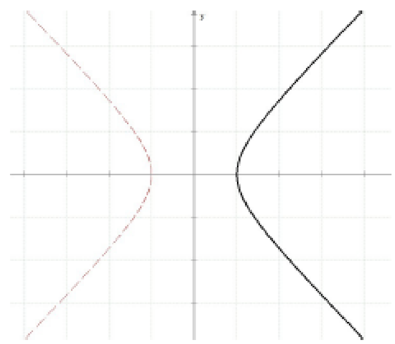
Tangent Lines to Parametric Curves

Example

Find the equation of the tangent line to the right branch of the hyperbola $x = \sec t, y = \tan t$, where $-\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $(\sqrt{2}, 1)$.

Solution: Note that the point $(\sqrt{2}, 1)$ corresponds to $t = \frac{\pi}{4}$. Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \csc t$$



Concavity

Example

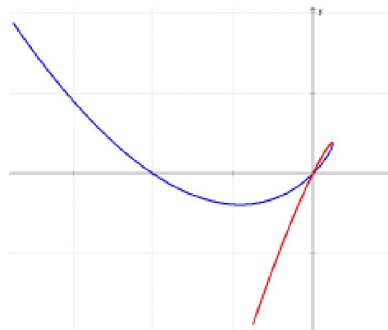
Find $\frac{d^2y}{dx^2}$ if $x = t - t^2, y = t - t^3$.

Solution: $\dfrac{dy}{dx} = \dfrac{1-3t^2}{1-2t}$

Thus,

$$\frac{d^2y}{dx^2} = \frac{\frac{d\left(\frac{1-3t^2}{1-2t}\right)}{dt}}{\frac{dx}{dt}} = \frac{\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}}{1-2t}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3}$$



Caution

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{-6t}{-2} = 3t \neq \frac{d^2y}{dx^2}.$$

Length of a Parametric Curve

Length of Arc

Let $C: x = x(t), y = y(t), a \leqslant t \leqslant b$, be a smooth curve traced exactly once. Then the length of C is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Proof. The length of arc of the smooth curve $y = F(x)$, from $t = a$ to $t = b$, is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Using differentials, we transform the integrand such that

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt} \cdot dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length

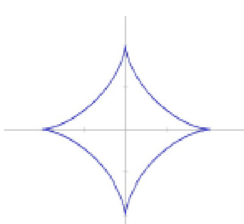
Example

Find the length of the *astroid*

$x = \cos^3 t$

$y = \sin^3 t$

$0 \leqslant t \leqslant 2\pi$



$$\begin{aligned} L &= 8\int_0^{\pi/4} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt \\ &= 8\int_0^{\pi/4} 3\cos t \sin t\, dt \\ &= 12\int_0^{\pi/4} \sin 2t\, dt \\ &= -6\cos 2t\Big|_0^{\pi/4} \\ &= 6 \end{aligned}$$

Arc Length

Example

Set-up the necessary definite integral that will solve the following.

1. Perimeter of the circle $x = 4\cos t$, $y = 4\sin t$.
2. Perimeter of the ellipse $x = 2 + 9\cos t$, $y = 1 - 4\sin t$.
3. Arclength of the curve $x = 4e^t$, $y = 5 - t^2$, with $t \in [1, 3]$.

Solution:

1. $\displaystyle\int_0^{2\pi} \sqrt{(-4\sin t)^2 + (4\cos t)^2}\, dt$
2. $\displaystyle\int_0^{2\pi} \sqrt{(-9\sin t)^2 + (-4\cos t)^2}\, dt$
3. $\displaystyle\int_1^{3} \sqrt{16e^{2t} + 4t^2}\, dt$

Exercises

1. Find the slope of the curve with parametric equations $x = \frac{1}{t^2}$, $y = t^3 - 5t$ at the point $(1, -4)$.

2. Find the concavity of the curve $x = t^2 - 4t,\ y = 5t^3 + t - 3$ when $t = 2$.

3. Set-up the necessary definite integral that will solve the following.

    1. Perimeter of the hypocycloid

    $$\begin{cases} x &= 2\cos t - 3\cos\left(\frac{2t}{3}\right) \\ y &= 2\sin t + 3\sin\left(\frac{2t}{3}\right) \end{cases}$$

    with $t \in [0, 2\pi]$.

    2. Arclength of the curve $y = x^3$ from $(-1, -1)$ to $(2, 8)$.