MATH 1020 WORKSHEET 10.1
Parametric Equations

If f and g are continuous functions on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations. The parametric equations along with the graph of (x,y) for all t in the interval make up the plane curve.

Sketch the curve represented by the parametric equations $x = 3t - 1$, $y = 2t + 1$ making sure to indicate the direction of motion. Write the corresponding rectangular equation by eliminating the parameter.

Solution. Note that a sketch is not provided, but direction of motion is indicated in the description below. Solving the x equation for t one has that $t = \dfrac{x+1}{3}$. Substituting this into the y equation the resulting rectangular equation is

$$y = \frac{2}{3}(x+1) + 1 = \frac{2}{3}x + \frac{5}{3}.$$

This is the equation of a line with slope $\frac{2}{3}$ with y-intercept of $\frac{5}{3}$ and x-intercept of $-\frac{5}{2}$. There are no restrictions given on t and there are no natural restrictions on t imposed by $f(t)$ or $g(t)$ so the sketch would include the entire line. The direction of motion in the rectangular plane is from bottom left to top right. This was determined by noting that a particle moves along the line from the point $(-1, 1)$ when $t = 0$ to the point $(2, 3)$ when $t = 1$.

Eliminate the parameter and write the corresponding rectangular equation for $x = 4 + 2\cos\theta$, $y = -1 + 4\sin\theta$.

Solution. To solve this type of parametric equations problem, one needs to solve for $\cos\theta$ and $\sin\theta$ in terms of x and y respectively and then substitute into the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.

$$x = 4 + 2\cos\theta \qquad\qquad y = -1 + 4\sin\theta$$
$$\tfrac{x-4}{2} = \cos\theta \qquad\qquad \tfrac{y+1}{4} = \sin\theta$$

Substituting into the trigonometric identity results in

$$\begin{aligned}\cos^\theta + \sin^2\theta &= 1\\ \left(\frac{x-4}{2}\right)^2 + \left(\frac{y+1}{4}\right)^2 &= 1\\ \frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} &= 1\end{aligned}$$

This equation is the equation of an ellipse with center at $(4, -1)$ and vertices at $(2, -1)$, $(6, -1)$, $(4, -5)$ and $(4, 3)$. Since there is no restriction given on θ and none exists from the definitions of $f(\theta)$ and $g(\theta)$, the entire ellipse is traced out.

The last step in our solution is to determine the direction of motion the curve is traced out. The chart on the right helps us to determine that the direction of motion is counterclockwise.

θ	x	y
0	6	-1
$\frac{\pi}{2}$	4	3
π	2	-1

Find two different sets of parametric equations for the line $y = 3x - 2$.

Solution. The simplest set of parametric equations for $y = 3x - 2$ is to set $x = t$ and then substitute that into the equation of the line to determine the resulting y equation. This gives

$$x = t \qquad y = 3t - 2$$

A second set of parametric equations can be formed by using some other function of t for x. However, care must be used to not introduce a restriction on the t-interval that would result in part of the line not being represented by the new set of parametric equations. One such second set of parametric equations that would work is

$$x = 5t \qquad y = 15t - 2$$

Another is the following

$$x = t + 3 \qquad y = 3t + 7$$

One that only gives you half of the line and therefore is not correct is

$$x = t^2 \qquad y = 3t^2 - 2$$

MATH 1020 WORKSHEET 10.2
Calculus with Parametric Curves

This sections included formulas for slope and concavity of parametric curves and for arc length when given parametric curves.

Find dy/dx and d^2y/dx^2 for the parametric equations $x = \sqrt{t}$, $\quad y = \sqrt{t-1}$ and evaluate each at $t = 2$.

Solution. For both the slope and concavity, one first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}$$

Next one applies the formula for the slope in parametric form

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{ft}} \\ &= \frac{\frac{1}{2\sqrt{t-1}}}{\frac{1}{2\sqrt{t}}} \\ &= \frac{\sqrt{t}}{\sqrt{t-1}} \\ &= \sqrt{\frac{t}{t-1}} \end{aligned}$$

Thus

$$\boxed{\left.\frac{dy}{dx}\right|_{t=2} = \sqrt{2}}.$$

To determine the concavity, we apply to parametric formula

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}\left(\sqrt{\frac{t}{t-1}}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{1}{2}\left(\frac{t}{t-1}\right)^{-1/2}\left(\frac{(t-1)-t}{(t-1)^2}\right)}{\frac{1}{2\sqrt{t}}} \\ &= 2\sqrt{t}\cdot\left(\frac{1}{2}\left(\frac{t}{t-1}\right)^{-1/2}\left(\frac{-1}{(t-1)^2}\right)\right) \end{aligned}$$

Evaluating at $t = 2$ one gets

$$\begin{aligned} \left.\frac{d^2y}{dx^2}\right|_{t=2} &= 2\sqrt{2}\cdot\left(\frac{1}{2}\left(\frac{2}{1}\right)^{-1/2}\left(\frac{-1}{1^2}\right)\right) \\ &= 2\sqrt{2}\cdot\left(\frac{1}{2}\frac{1}{\sqrt{2}}(-1)\right) \\ &= -1 \end{aligned}$$

Thus

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{t=2} = -1}.$$

Find the arc length of the curve given by $x = t$, $y = \frac{t^5}{10} + \frac{1}{6t^3}$ on the interval $1 \leq t \leq 2$.

Solution. One first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$$

Next one applies the formula for the arc length in parametric form

$$\begin{aligned}
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt \\
&= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - 2 \cdot \frac{t^4}{2} \cdot \frac{1}{2t^4} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - \frac{1}{2} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{\left(\frac{t^8}{4} + \frac{1}{2} + \frac{1}{4t^8}\right)}\, dt \\
&= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\, dt \\
&= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right)\, dt \\
&= \left(\frac{t^5}{10} - \frac{1}{6t^3}\right)\Bigg|_1^2 \\
&= \left(\frac{32}{10} - \frac{1}{10}\right) - \left(\frac{1}{48} - \frac{1}{6}\right) \\
&= \frac{31}{10} - \frac{-7}{8} \\
&= \frac{744 + 35}{240}
\end{aligned}$$

Thus

$$\boxed{\int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt = \frac{779}{24}}.$$