Math 113 Final Exam Practice

The Final Exam is comprehensive. You should refer to prior reviews when studying material in chapters 6, 7, 8, and 11. This review will cover chapter 10. This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section gives the answers of the questions in section 2.

Review

10.1 Parametric Equations

We learned how to define curves parametrically. That is, we learned how to describe a curve given by an equation

$$H(x, y) = 0$$

in terms of a pair of functions

$$x = f(t),\ y = g(t).$$

You will need to be able to do the following:

(a) Graph a curve from its parametric equations.

(b) Recognize the curve of a set of parametric equations.

(c) Eliminate the parameter of the parametric equations to find an equation in x and y describing the curve.

(d) Construct a set of parametric equations for a curve written in cartesian coordinates.

10.2 Calculus of Parametric Equations

In the discussion below, we will assume that a curve can be described parametrically by

$$x = f(t),$$
$$y = g(t).$$

Slopes

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$

This gives a formula for the slope as a function of parameter.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Arclength

$$s = \int_{t_0}^{t_1} \sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$

Surface Area

Rotated about the x axis:

$$S = \int_{t_0}^{t_1} 2\pi f(x)\sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$

Rotated about the y axis:

$$S = \int_{t_0}^{t_1} 2\pi g(x)\sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$

Area under the curve

Area between the curve and the x axis:

$$A = \int_{t_0}^{t_1} y\, dx = \int_{t_0}^{t_1} g(t)f'(t)\, dt$$

Area between the curve and the y axis:

$$A = \int_{t_0}^{t_1} x\, dy = \int_{t_0}^{t_1} f(t)g'(t)\, dt$$

10.3 Polar Coordinates

In this section, we learned how to write points and equations in polar coordinates.

You will need to be able to do the following:

(a) Convert a point from cartesian coordinates to polar coordinates and vice-versa.

(b) Be able to write a polar curve in cartesian coordinates and a cartesian curve in polar coordinates.

(c) Be able to graph and recognize polar curves.

You will need to know the following formulas:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \tan^{-1}(y/x)$ if (x, y) is in quadrants 1 or 4. Otherwise, $\theta = \tan^{-1}(y/x) + \pi$.

You will also need to be able to find the slope of a polar curve. Fortunately, we can do this using the standard parameterization of a polar curve. If $r = f(\theta)$ is a polar curve, then from the above equations we can write

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta.$$

Then we can use the techniques of section 10.2 to find the slope of the tangent line.

10.4 Calculus of Polar coordinates

Note that we can use the parameterization of polar curves mentioned in the previous section to find arclength also. However, in this case, the formula simplifies considerably, so it is better to use the simplified formula directly. If $r = f(\theta)$, then the arclength is given by

$$s = \int_{\theta_0}^{\theta_1} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

We also wish to find area underneath polar curves. However, since polar curves are defined by angle, underneath really translates to "between the curve and the origin". The area "inside" a polar curve $r = f(\theta)$, or between the polar curve and the origin is given by

$$A = \int_{\theta_0}^{\theta_1} \frac{1}{2} f^2(\theta)\, d\theta.$$

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10.5 Conic Sections

Parabolas

$$4d(y-k) = (x-h)^2$$

Vertex is at (h,k), focus is at $(h, k+d)$, directrix is at $y = k - d$.

$$4d(x-h) = (y-k)^2$$

Vertex is at (h,k), focus is at $(h+d, k)$, directrix is at $x = k - d$.

Ellipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$c^2 = |a^2 - b^2|$$

The ellipse is horizontal if $a > b$ and vertical if $b > a$ and a circle if $a = b$.
Center: (h,k), vertices at $(h-a,k)$, $(h+a,k)$, $(h,k-b)$, $(h,k+b)$.
If $a > b$ the foci are $(h-c,k)$, $(h+c,k)$.
If $a < b$ the foci are $(h,k-c)$, $(h,k+c)$.

Hyperbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$c^2 = a^2 + b^2$$

Center: (h,k) Vertices $(h-a,k)$, $(h+a,k)$. Foci: $(h-c,k)$, $(h+c,k)$ Asymptotes: $y - k = \pm\frac{b}{a}(x-h)$

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$
$$c^2 = a^2 + b^2$$

Center: (h,k) Vertices $(h,k-b)$, $(h,k+b)$. Foci: $(h,k-c)$, $(h,k+c)$ You may need to determine the properties of a conic by completing the square.

Questions

Try to study the review notes and memorize any relevant equations before trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

In problems 1 to 3, graph the parametric curve.

1. $x(t) = \cos t$, $y(t) = \sin(2t)$.
2. $x(t) = e^{2t}$, $y(t) = \ln(t) + 1$.
3. $x(t) = \sqrt{t}$, $y(t) = t^{3/2} - 2t$.

   In problems 4 to 6, eliminate the parameter to find a Cartesian equation of the curve.

4. $x(t) = \cos t$, $y(t) = \sin(2t)$.
5. $x(t) = e^{2t}$, $y(t) = \ln(t) + 1$.
6. $x(t) = \sqrt{t}$, $y(t) = t^{3/2} - 2t$.

   In problems 7 to 8, find parametric equations for the curve

7. $x^2 + \frac{y^2}{4} = 1$
8. $y = x^2 + 2x - 1$
9. Find an equation of the tangent to the curve at the given point.

   $x = \cos(3\theta)+\sin(2\theta), \quad y = \sin(3\theta)+\cos(2\theta); \quad \theta = 0$

10. For which values of t is the tangent to curve horizontal or vertical? Determine the concavity of the curve.

    $$x = t^2 - t - 1, \quad y = 2t^3 - 6t - 1$$

11. Find the area enclosed by the curve $x = t^2 - 2t, y = \sqrt{t}$ and the y-axis.
12. Find the area of one quarter of the ellipse described by $x = 5\sin(t), y = 2\cos(t)$.
13. Find the exact length of the curve: $x = \frac{t}{1+t}, y = \ln(1+t); \quad 0 \le t \le 2$.
14. Find the exact length of the curve: $x = e^t + e^{-t}, y = 5 - 2t; \quad 0 \le t \le 3$.
15. Find the exact surface area by rotating the curve about the x-axis: $x = t^3, y = t^2; \quad 0 \le t \le 1$.
16. The Cartesian coordinates for a point are $(-1, -\sqrt{3})$. Find polar coordinates (r, θ) for the point where $r > 0$ and $0 \le \theta < 2\pi$.
17. Find the distance between the points with polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$.
18. Find a polar equation for the curve represented by the Cartesian equation $x^2 + y^2 = 9$.
19. Identify the curve given in polar coordinates by $r = 4\sin\theta$ by finding a Cartesian equation for the curve.
20. Graph in polar coordinates $r = 2\cos(3\theta)$.
21. Graph in polar coordinates $r = \sin(2\theta)$.
22. Graph in polar coordinates $r = 1 + \cos\theta$.
23. Graph in polar coordinates $r^2 = \cos(2\theta)$.
24. Find the area inside the circle $r = 6\sin\theta$ and outside the limaçon $r = 2 + 2\sin\theta$.
25. Find the area of one petal of the rose given by $r = \cos 3\theta$.
26. Find the length of the polar curve given by $r = \theta$ for $\theta \in [0, \pi]$.
27. Set up but do not evaluate an integral for the length of the polar curve given by $r = \theta + \sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$.

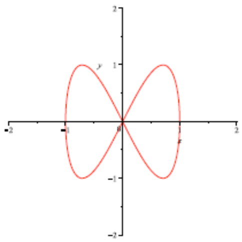
    In problems 28 to 31, give the equation of the indicated conic section in standard form.

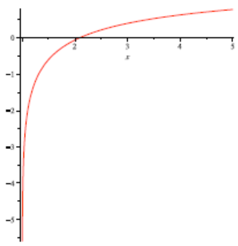
28. Parabola: Vertex at (2,-3), directrix at $x = 0$.
29. Parabola: Focus at (1,-4), directrix at $y = -5$.
30. Ellipse: Focus at (2,6), vertices at (2,9) and (2,0).
31. Hyperbola: Vertices at (0,2) and (0,10), focus at (0,12)

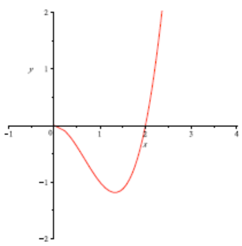
    In problems 32 to 34, write the conic section in standard form. Identify the conic section, and give the center, vertices, focus or foci, and asymptotes (if applicable).

32. $4x^2 - 40x + 64 - 9y^2 - 36y - 144 = 0$
33. $x = \frac{1}{12}y^2 - \frac{1}{2}y + 7$
34. $x^2 + 2x + 9y^2 - 54y + 73 = 0$

Answers

1. 

2. 

3. 

4. $y = 2x\sqrt{1-x^2}$

5. $y = \ln(\ln(x)) - \ln(2) + 1$

6. $y = x^3 - 2x^2$

7. $x = \cos(t)$, $y = 2\sin(t)$

8. $x = t$, $y = t^2 + 2t - 1$

9. $y = \frac{3}{2}x - \frac{1}{2}$

10. vertical at $t = 1/2$, horizontal at ± 1, concave up when $t > 1/2$, concave down when $t < 1/2$.

11. $\frac{8\sqrt{2}}{15}$

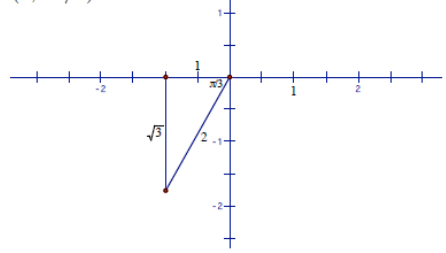
12. $\frac{5\pi}{2}$

13. $-\sqrt{10}/3 + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})$

14. $e^3 - e^{-3}$

15. $\frac{2}{1215}\pi(247\sqrt{13} + 64)$.

16. $(2, 4\pi/3)$



17. Convert $(2, \pi/3)$ to Cartesian coordinates. $x = 2\cos(\pi/3) = 1$, $y = 2\sin(\pi/3) = \sqrt{3}$

    Convert $(4, 2\pi/3)$ to Cartesian coordinates. $x = 4\cos(2\pi/3) = -2$, $y = 4\sin(2\pi/3) = 2\sqrt{3}$

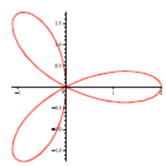
    Find the distance between $(1, \sqrt{3})$ and $(-2, 2\sqrt{3})$ in Cartesian coordinates.

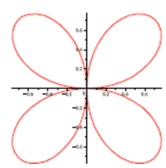
    $\sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

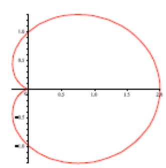
18. $r = 3$.

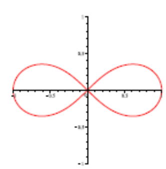
19. $r = 4\sin\theta$ gives $r^2 = 4r\sin\theta$. In Cartesian coordinates $x^2 + y^2 = 4y$ or $x^2 + y^2 - 4y + 4 = 4$ or $x^2 + (y-2)^2 = 4$

    This is a circle of radius 2 centered at (0,2).

20. 

21. 

22. 

23. 

24. $r = 6\sin\theta$ and $r = 2 + 2\sin\theta$ intersect at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. Use symmetry to get the integral:

$$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}\left((6\sin\theta)^2 - (2+2\sin\theta)^2\right)\, d\theta = 4\pi$$

25. One petal is drawn for $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$. Use symmetry to get the integral:

$$A = 2\int_0^{\pi/6} \frac{1}{2}\cos^2 3\theta \, d\theta = \frac{\pi}{12}$$

26. The length of the curve is the integral:

$$L = \int_0^{\pi} \sqrt{\theta^2+1}\, d\theta = \frac{1}{2}\pi\sqrt{\pi^2+1} + \frac{1}{2}\ln\left(\pi + \sqrt{\pi^2+1}\right)$$

27. The length of the curve is the integral:

$$L = \int_0^{\pi/2} \sqrt{(\theta+\sin\theta)^2 + (1+\cos\theta)^2}\, d\theta$$

28. $8(x-2) = (y+3)^2$

29. $2(y+\frac{9}{2}) = (x-1)^2$

30. $\dfrac{(x-2)^2}{81} + \dfrac{(y-6)^2}{\frac{81}{4}} = 1$

31. $\dfrac{(y-6)^2}{16} - \dfrac{x^2}{20} = 1$

32. $\dfrac{(x-5)^2}{36} - \dfrac{(y+2)^2}{16} = 1$

33. $12(x-25/4) = (y-3)^2$

34. $\dfrac{(x+1)^2}{9} + (y-3)^2 = 1$