MATH 1020 WORKSHEET 10.3
Polar Coordinates and Polar Graphs

Polar coordinates require the basic transformation equations

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = y/x$$

Given the polar point $(-1, 5\pi/4)$, find the corresponding Cartesian coordinates for the point.

Solution. We apply the transformation equations to determine the x and y coordinates of the point.

$$x = -1 \cdot \cos\left(\tfrac{5\pi}{4}\right) \qquad y = -1 \cdot \sin\left(\tfrac{5\pi}{4}\right)$$
$$x = -1\left(\tfrac{-1}{\sqrt{2}}\right) \qquad y = -1\left(\tfrac{-1}{\sqrt{2}}\right)$$
$$x = \tfrac{1}{\sqrt{2}} \qquad y = \tfrac{1}{\sqrt{2}}$$

Thus, writing the answer as a point we have

$$\boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}.$$

Convert the polar equation $r = 4\sin\theta$ to a Cartesian form and sketch its graph.

Solution. Transforming to a polar equation, it is best to start by replacing any trigonometric functions in the equation before regrouping and then replacing any r terms.

$$\begin{aligned} r &= 4\sin\theta \\ &= 4\left(\frac{y}{r}\right) \\ r^2 &= 4y \\ x^2 + y^2 &= 4y \end{aligned}$$

From here, the easiest way to sketch the equation is to bring the $4y$ over to the LHS and complete the square so that the Cartesian equation is in a familiar form.

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + \left(y^2 - 4y + \_\right) &= 0 + \_ \\ x^2 + \left(y^2 - 4y + \underline{4}\right) &= 0 + \underline{4} \\ x^2 + (y-2)^2 &= 4 \end{aligned}$$

One recognizes that we have the equation of a circle. Thus

$$\boxed{x^2+y^2=4y \text{ is a circle with center at } (0,2) \text{ and radius } r=2}.$$

Convert the Cartesian equation $xy=4$ to a polar form.

Solution. Converting a Cartesian equation into polar form is done by substituting the transformation equations for $x=r\cos\theta$ and $y=r\sin\theta$ into the given equation.

$$\boxed{r^2\cos\theta\sin\theta=4}.$$

Find the slope of the tangent line to the polar curve $r=2-\sin\theta$ at $\theta=\frac{\pi}{3}$.

Solution. To determine slope of the tangent line to the polar curve, we must first convert the polar equations into parametric form using the transformation equations. For parametric form we have

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ x &= (2-\sin\theta)\cos\theta & y &= (2-\sin\theta)\sin\theta \\ x &= 2\cos\theta-\cos\theta\sin\theta & y &= 2\sin\theta-\sin^2\theta \end{aligned}$$

Now we need to determine $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$.

$$\frac{dx}{d\theta}=-2\sin\theta-\cos^2\theta+\sin^2\theta \qquad \frac{dy}{d\theta}=2\cos\theta-2\sin\theta\cos\theta$$

Substituting into the parametric formula for slope we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{2\cos\theta-2\sin\theta\cos\theta}{-2\sin\theta-\cos^2\theta+\sin^2\theta} \end{aligned}$$

Now we can evaluate at $\theta=\pi/3$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\theta=\pi/3} &= \left.\frac{2\cos\theta-2\sin\theta\cos\theta}{-2\sin\theta-\cos^2\theta+\sin^2\theta}\right|_{\theta=\pi/3} \\ &= \frac{2\cos(\pi/3)-2\sin(\pi/3)\cos(\pi/3)}{-2\sin(\pi/3)-\cos^2(\pi/3)+\sin^2(\pi/3)} \\ &= \frac{2\left(\frac{1}{2}\right)-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{-2\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1-\frac{\sqrt{3}}{2}}{-\sqrt{3}-\frac{1}{4}+\frac{3}{4}} \\ &= \frac{1-\frac{\sqrt{3}}{2}}{-\sqrt{3}+\frac{1}{2}} \end{aligned}$$

Thus the slope of the tangent line to the polar curve $r=2-\sin\theta$ at $\theta=\frac{\pi}{3}$ is

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=\pi/3}=\frac{2-\sqrt{3}}{-2\sqrt{3}+1}}.$$