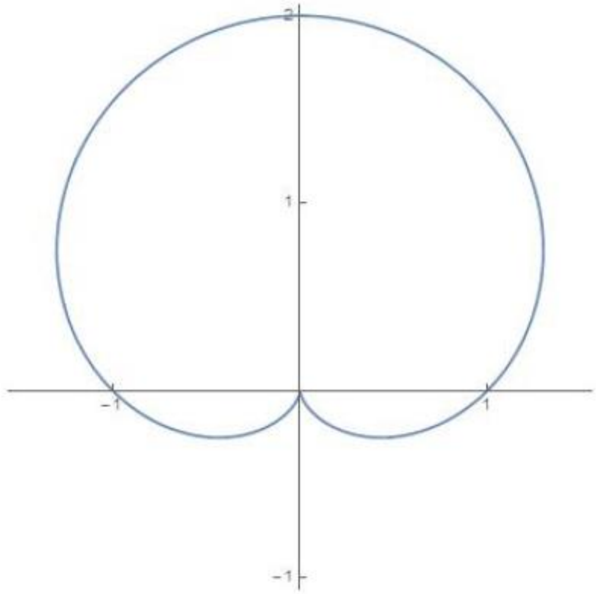
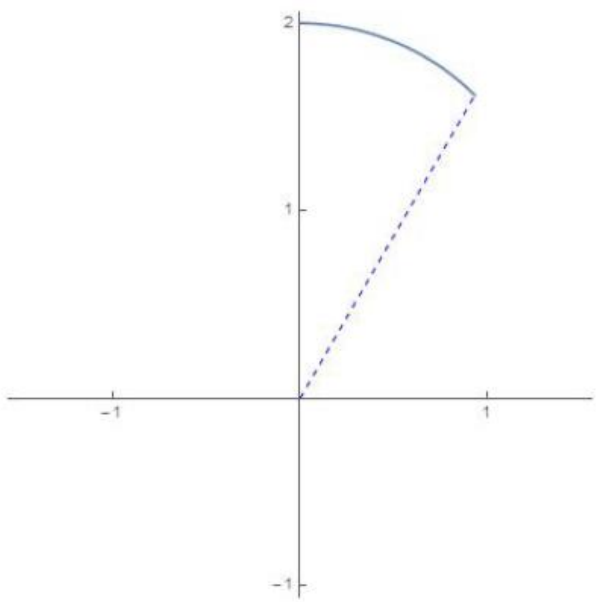
Area in Polar Coordinates

The curve $r = 1 + \sin\theta$ is graphed below:

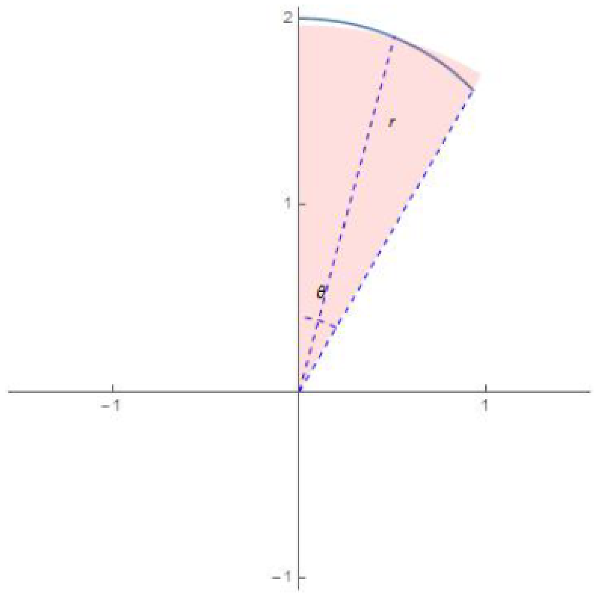


The curve encloses a region whose area we would like to be able to determine. In this section we will find a formula for determining the area of regions bounded by polar curves; to do this, we again make use of the idea of approximating a region with a shape whose area we can find, then use calculus to make the approximations exact.

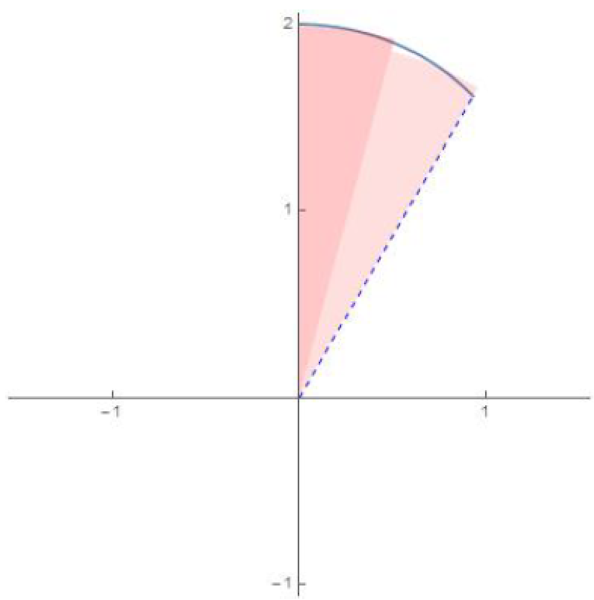
To understand the process, consider the area of the region enclosed "between" the polar curve below and the origin:



We can imagine overlaying the region with a sector of a circle (whose area we can readily find using the formula $A = 1/2r^2\theta$):



We can use more sectors (i.e., decrease the sector's angle θ) to get a better approximation:



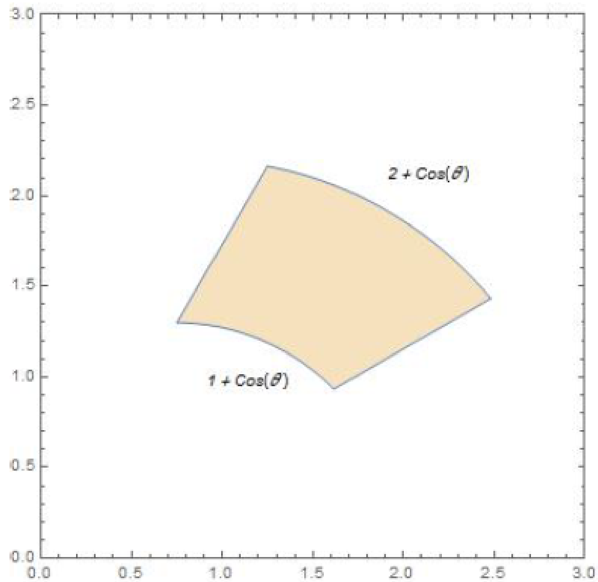
Again, the area of a sector of a circle of radius r is given by

$$A = \frac{1}{2}r^2\theta,$$

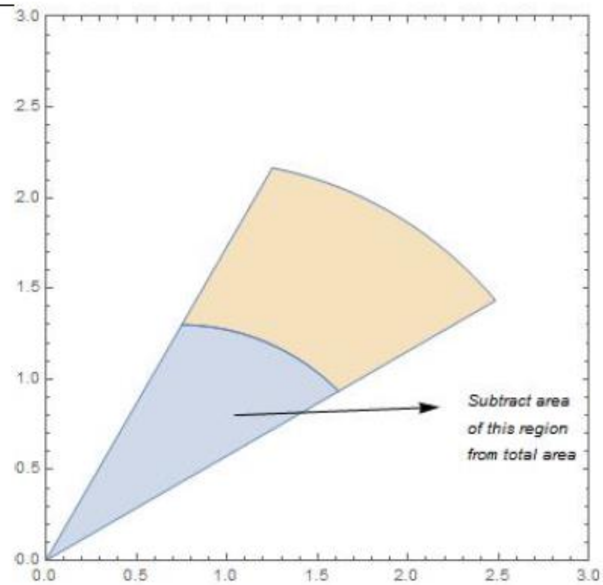
where θ is the central angle of the sector; if the region whose area we wish to find is bounded between $\theta = a$ and $\theta = b$, then the area between the curve and the origin is

$$A = \frac{1}{2}\int_a^b r^2 d\theta.$$

We can use a similar idea to find the area of a region enclosed between two polar curves; for instance, suppose that we wished to determine the area between $r_1 = 2 + \cos\theta$ and $r_2 = 1 + \cos\theta$ in the graphic below (shaded in brown):



As with finding areas of regions between Cartesian curves, the easiest way to solve this problem is to find the area bounded between the outer curve and the origin, then subtract the area between the inner curve and the origin. In the graphic below, we could find the total area bounded between $r1$ and the origin (area of the blue and brown regions), and then subtract the total area bounded between $r2$ and the origin (area of the blue region).



So the area of the region bounded between $r_2(\theta)$ and $r_1(\theta)$, where $r_1(\theta) \geq r_2(\theta)$ for $a \leq \theta \leq b$ is given by

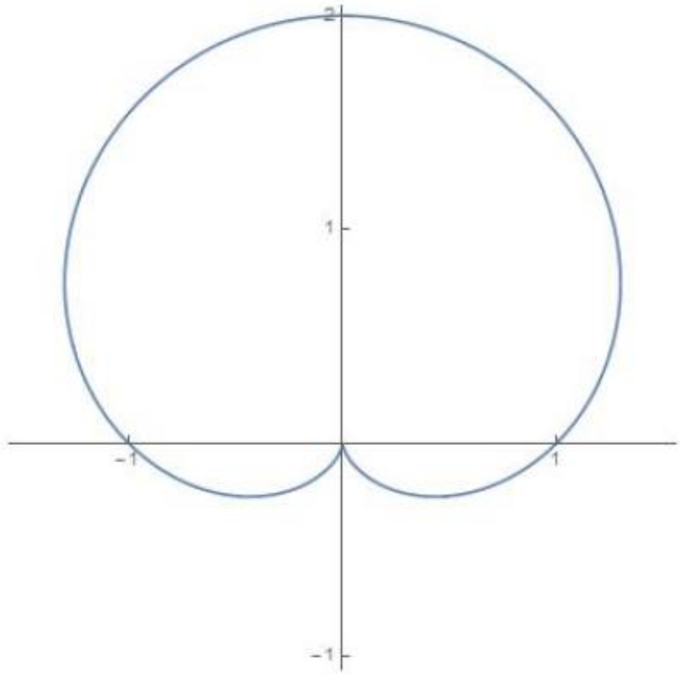
$$A = \frac{1}{2} \int_a^b r_1^2 - r_2^2 \, \mathrm{d}\theta.$$

To determine the area of a region described by a polar equation, follow these steps:

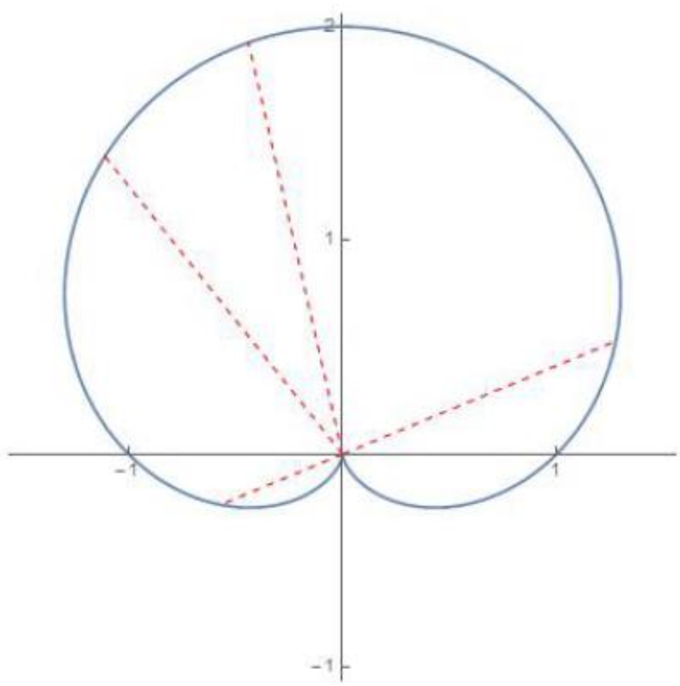
1. Graph the region; draw lines from the origin to the curve to determine r.
2. If necessary, determine where the curves intersect.
3. Set up an integral for each region with distinct outer or inner boundaries.
4. Determine the bounds of integration for each region by determining where θ "starts" and "stops".
5. Set up and evaluate the resulting integral(s).

Example. Find the area enclosed by the cardioid $r = 1 + \sin\theta$.

The graph of the curve is below:



Notice that lines from the origin through the regiona always end at $r = 1 + \sin\theta$:



In addition, the lines sweep through the region from $\theta = 0$ to $\theta = 2\pi$, so the area is given by

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \sin^2\theta d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \frac{1}{2}(1-\cos(2\theta))d\theta \\
&= \frac{1}{2}(\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin(2\theta))\Big|_0^{2\pi} \\
&= \frac{1}{2}(\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin(2\theta))\Big|_0^{2\pi} \\
&= \frac{1}{2}(3\pi - 2 + 2) \\
&= \frac{3\pi}{2}.
\end{aligned}$$

Example. Find the area of the region in inside of $r = 2 - 2\cos\theta$ and outside of $r = 2$.

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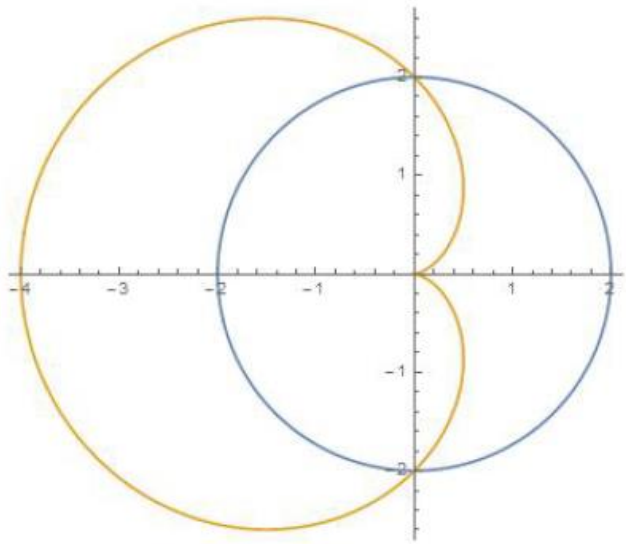
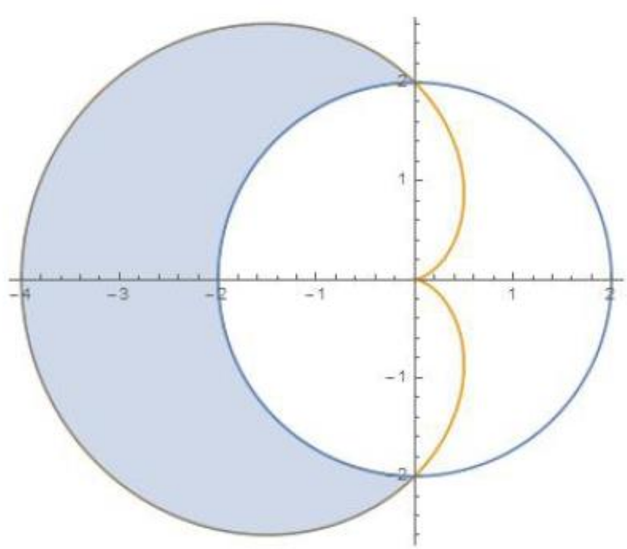
The polar curves are plotted below:



The region whose area we wish to find is shaded below:

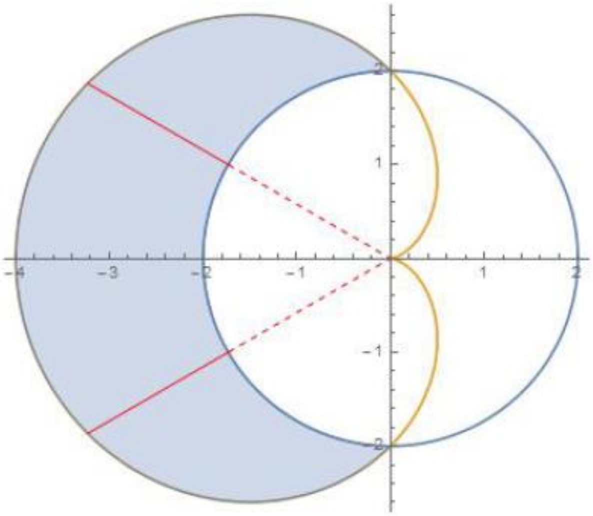


We need to determine where the two curves intersect, i.e. where $2 - 2\cos\theta = 2$. Thus we solve

$$\begin{aligned} 2 - 2\cos\theta = 2 \;&\Rightarrow\; \cos\theta = 0 \\ &\Rightarrow\; \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}. \end{aligned}$$

So the first point of intersection in the graph is $\frac{\pi}{2}$, and the second is $\frac{3\pi}{2}$.

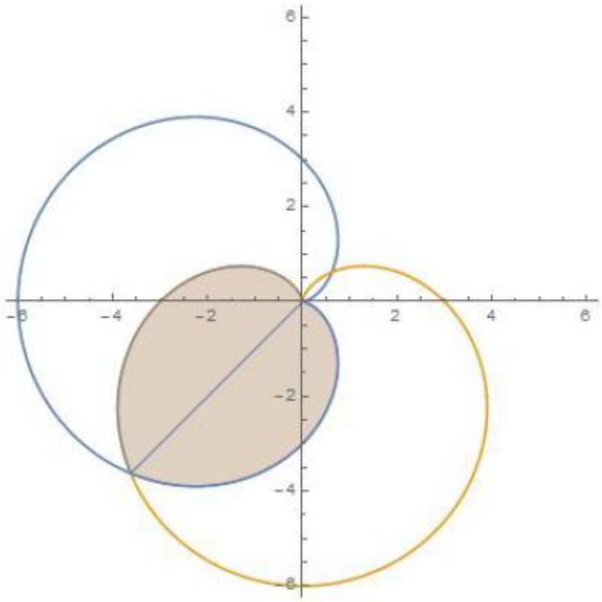
Notice that, as we draw radii from the origin through the region of interest, those radii enter the region through $r = 2$ and exit it at $r = 2 - 2\cos\theta$:



The regions' area can be found by determining the area enclosed by $2 - 2\cos\theta = 2$ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and subtracting off the area enclosed by $r = 2$ on the same bounds. So the area is given by

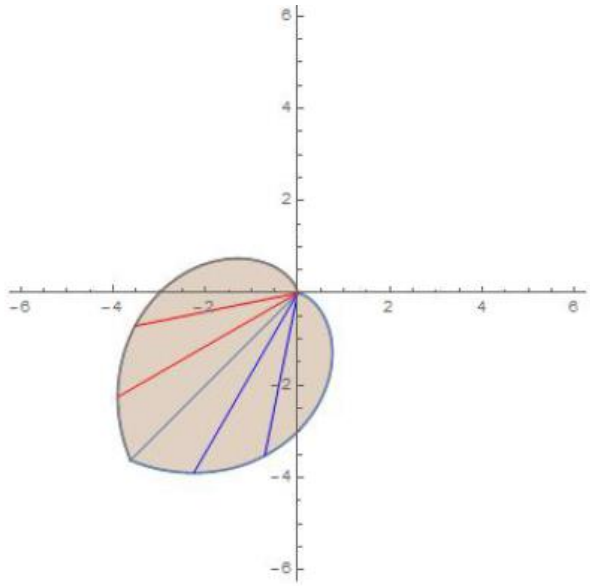
$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 - 2\cos\theta)^2 - (2)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta - 4 \, \mathrm{d}\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos^2\theta - 4\cos\theta \, \mathrm{d}\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 + \cos(2\theta) - 4\cos\theta \, \mathrm{d}\theta \\
&= \theta + \frac{1}{2}\sin(2\theta) - 4\sin\theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&= \frac{3\pi}{2} + 4 - \frac{\pi}{2} + 4 \\
&= \pi + 8.
\end{aligned}
$$

Example. The curves $r = 3 - 3\cos\theta$ and $r = 3 - 3\sin\theta$ are graphed below:



Set up an expression for the area of the region shaded above.

It will be helpful to begin this problem by thinking about radii through the region of interest; we graph several such radii below:



Notice that there are *two different* types of radii: those that exit the region through $3 - 3\sin\theta$ (in red), and those that exit through $3 - 3\cos\theta$ (in blue). Thus we need two different integrals in order to write an expression for the area of the region.

The radii that end at $r = 3 - 3\sin\theta$ start at $\theta = \pi/2$ and end at $\theta = 5\pi/4$; on the other hand, the radii that end at $r = 3 - 3\cos\theta$ start at $\theta = 5\pi/4$ and end at $\theta = 2\pi$. Thus the expression for the area is

$$A = \frac{1}{2}\int_{\pi/2}^{5\pi/4} (3 - 3\sin\theta)^2 \, \mathrm{d}\theta + \frac{1}{2}\int_{5\pi/4}^{2\pi} (3 - 3\cos\theta)^2 \, \mathrm{d}\theta.$$