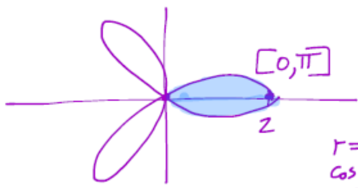
Worksheet 8.2—Polar Area
Show all work. **Calculator permitted** except unless specifically stated.

Short Answer: Sketch a graph, shade the region, and find the area.

(No Calculator)

1. one petal of $r = 2\cos(3\theta)$



$r = 0$
$\cos 3\theta = 0$
$3\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{6}$

Rose Curve

$$\text{Area} = (2)\left(\tfrac{1}{2}\right)\int_0^{\pi/6} \left(2\cos(3\theta)\right)^2 d\theta$$

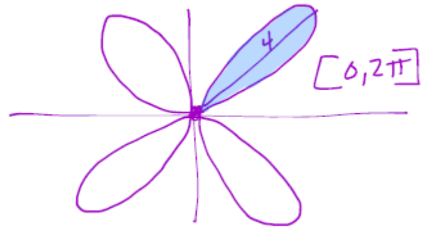
symm

By Hand:

$$= (4)\int_0^{\pi/6} \tfrac{1}{2}(1+\cos(6\theta))\, d\theta$$
$$= 2\left[\theta + \tfrac{1}{6}\sin(6\theta)\right]\Big|_0^{\pi/6}$$
$$= 2\left[\left(\tfrac{\pi}{6} + \tfrac{1}{6}\sin\pi\right) - (0 + \sin 0)\right]$$
$$= 2\left(\tfrac{\pi}{6}\right)$$
$$= \tfrac{\pi}{3}$$

(No Calculator)

2. one petal of $r = 4\sin(2\theta)$



$r = 0$
$4\sin 2\theta = 0$
$\sin 2\theta = 0$
$2\theta = 0$
$2\theta = \pi$
$\theta = 0$
$\theta = \frac{\pi}{2}$
consecutive polar zeros

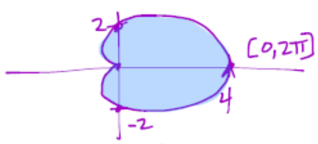
Rose Curve

$$\text{Area} = \tfrac{1}{2}\int_0^{\pi/2} (4\sin 2\theta)^2\, d\theta$$

by Hand:

$$= \tfrac{1}{2}(16)\int_0^{\pi/2} \left(\tfrac{1}{2}(1-\cos 4\theta)\right) d\theta$$
$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]\Big|_0^{\pi/2}$$
$$= 4\left[\left(\tfrac{\pi}{2} - \sin 2\pi\right) - (0 - 0)\right]$$
$$= 2\pi$$

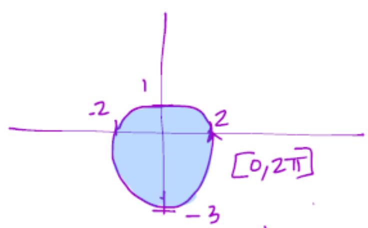
3. interior of $r = 2 + 2\cos\theta$
(no calculator)



Cardioid

$$\text{Area} = \tfrac{1}{2}\int_0^{2\pi} (2+2\cos\theta)^2\, d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} (4 + 8\cos\theta + 4\cos^2\theta)\, d\theta$$
$$= \left(\tfrac{1}{2}\right)(4)\int_0^{2\pi} \left(1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right) d\theta$$
$$= 2\int_0^{2\pi} \left(\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta$$
$$= 2\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]\Big|_0^{2\pi}$$
$$= 2\left[(3\pi + 0 + 0) - (0 + 0 + 0)\right]$$
$$= 6\pi$$

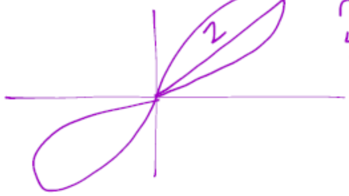
4. interior of $r = 2 - \sin\theta$
(no calculator)



dimpled cardioid/limaçon

$$\text{Area} = \tfrac{1}{2}\int_0^{2\pi} (2-\sin\theta)^2\, d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} (4 - 4\sin\theta + \cos^2\theta)\, d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left(4 - 4\sin\theta + \tfrac{1}{2}(1+\cos 2\theta)\right) d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{9}{2} - 4\sin\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta$$
$$= \tfrac{1}{2}\left[\left(\tfrac{9}{2}\theta + 4\cos\theta + \tfrac{1}{4}\sin 2\theta\right)\right]\Big|_0^{2\pi}$$
$$= \tfrac{1}{2}\left[(9\pi + 4 + 0) - (0 + 4 + 0)\right] = \tfrac{9\pi}{2}$$

5. interior of $r^2 = 4\sin(2\theta)$



$r^2 = 0$

$4\sin 2\theta = 0$

$\begin{cases} 2\theta = 0 \\ 2\theta = \pi \end{cases}$

$\begin{cases} \theta = 0 \\ \theta = \frac{\pi}{2} \end{cases}$

lemniscate

$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\, d\theta$

$= 2\int_0^{\pi/2} \sin 2\theta\, d\theta$

$= -2(\frac{1}{2})\cos 2\theta \Big|_0^{\pi/2}$

$= -[\cos\pi - \cos 0]$

$= -[-1-1]$

$= 2$ (one petal)

so, total area (of both petals) is $2(2) = 4$

6. inner loop of $r = 1 + 2\cos\theta$

$[0, 2\pi]$ limaçon

between consecutive polar zeros.

$r = 0$

$1 + 2\cos\theta = 0$

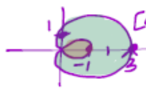
$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\text{Area} = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2\, d\theta$

Calculator: $\text{Area} = \pi - \frac{3\sqrt{3}}{2} \approx 0.543$ or 0.544

7. between the loops of $\underbrace{r = 1 + 2\cos\theta}_{\text{Same as in \#6}}$



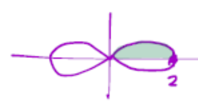
$[0, 2\pi]$ limaçon

$\text{Area} = \underset{\text{double}}{2}\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta - \frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= \pi + 3\sqrt{3}$

≈ 8.337 or 8.338

8. one loop of $r^2 = 4\cos(2\theta)$



use symmetry: double from $\theta = 0$ + 1st polar zero

$r^2 = 0$

$4\cos(2\theta) = 0$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

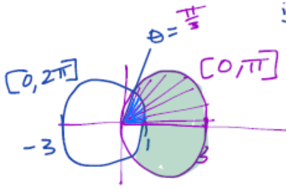
$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/4} 4\cos(2\theta)\, d\theta\right]$

$= 4(\frac{1}{2})\sin(2\theta)\Big|_0^{\pi/4}$

$= 2[\sin\frac{\pi}{2} - \sin 0]$

$= 2$

9. inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$



intersect

$3\cos\theta = 2 - \cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/3}(3\cos\theta)^2\,d\theta - \frac{1}{2}\int_0^{\pi/3}(2-\cos\theta)^2\,d\theta\right]$$

Symmetry

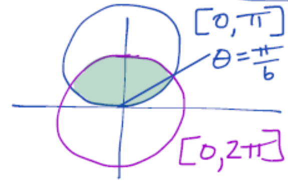
$$= \int_0^{\pi/3}\left[9\cos^2\theta - (2-\cos\theta)^2\right]d\theta$$

*can put as 1 integral since the same interval

$$= 3\sqrt{3}$$

$$\approx 5.196$$

10. common interior of $\boxed{r = 4\sin\theta}$ and $\boxed{r = 2}$



intersect

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/6}(4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(2)^2\,d\theta\right]$$

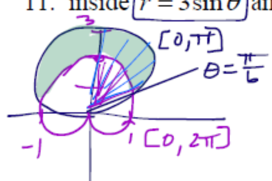
Symmetry

$$= \int_0^{\pi/6}(16\sin^2\theta)\,d\theta + \int_{\pi/6}^{\pi/2}4\,d\theta$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

$$\approx 4.913$$

11. inside $\boxed{r = 3\sin\theta}$ and outside $\boxed{r = 1 + \sin\theta}$



intersect

$3\sin\theta = 1 + \sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$$\text{Area} = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}\left((3\sin\theta)^2 - (1+\sin\theta)^2\right)d\theta\right]$$

Symmetry

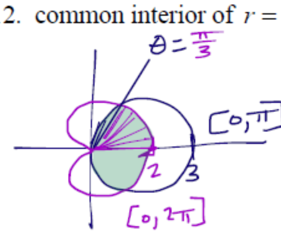
*can put as 1 integral since the same interval

$$= \int_{\pi/6}^{\pi/2}\left(9\sin^2\theta - (1+\sin\theta)^2\right)d\theta$$

$$= \pi$$

$$\approx 3.141 \text{ or } 3.142$$

12. common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$



intersect

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

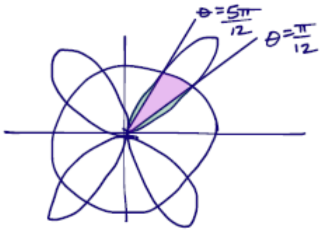
$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta\right]$$

$$= \int_0^{\pi/3}(1+\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2}9\cos^2\theta\,d\theta$$

$$= \frac{5\pi}{4}$$

$$\approx 3.926 \text{ or } 3.927$$

13. common interior of $r = 4\sin(2\theta)$ and $r = 2$



intersect

$4\sin(2\theta) = 2$

$\sin(2\theta) = \frac{1}{2}$

$\begin{cases} 2\theta = \frac{\pi}{6} \\ 2\theta = \frac{5\pi}{6} \end{cases}$

$\theta = \frac{\pi}{12}$

$\theta = \frac{5\pi}{12}$

Find 1 sliver, then multiply by 4 petals

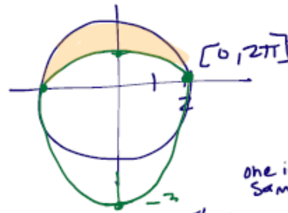
$$\text{Area} = 4\left[2 \cdot \frac{1}{2}\int_0^{\pi/12} (4\sin(2\theta))^2\, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\, d\theta\right]$$

$$= 4\int_0^{\pi/12} 16\sin^2(2\theta)\, d\theta + \int_{\pi/12}^{5\pi/12} 8\, d\theta$$

$$= 64\int_0^{\pi/12} \sin^2 2\theta\, d\theta + \int_{\pi/12}^{5\pi/12} 8\, d\theta$$

$$= 9.826 \text{ or } 9.827$$

14. inside $r = 2$ and outside $r = 2 - \sin\theta$



one integral since same interval

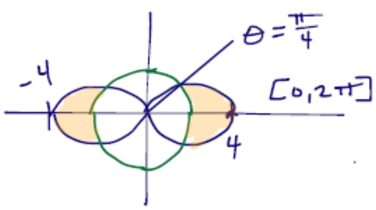
$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/2} \left((2)^2 - (2-\sin\theta)^2\right) d\theta\right]$$

$$= \int_0^{\pi/2} \left(4 - (2-\sin\theta)^2\right) d\theta$$

$$= 4 - \frac{\pi}{4}$$

$$\approx 3.214 \text{ or } 3.215$$

15. inside $r = 2 + 2\cos(2\theta)$ and outside $r = 2$



intersect

$2 + 2\cos 2\theta = 2$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

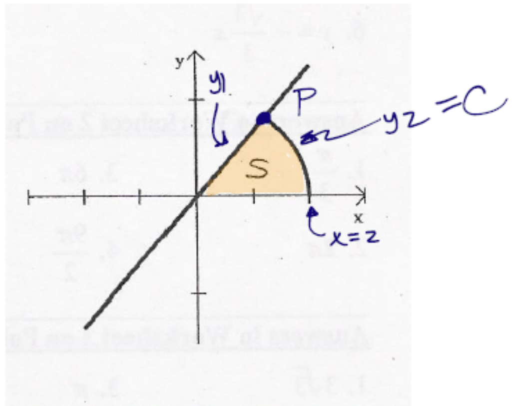
$$\text{Area} = 4\left[\frac{1}{2}\int_0^{\pi/4} \left[(2+2\cos 2\theta)^2 - (2)^2\right] d\theta\right]$$

$$= 2\int_0^{\pi/4} \left((2+2\cos 2\theta)^2 - 4\right) d\theta$$

$$= 11.5707$$

Free Response

16. The figure shows the graphs of the line $y = \frac{2}{3}x$ $= y1$ and the curve C given by $y = \sqrt{1 - \frac{x^2}{4}}$ $= y2$. Let S be the region in the first quadrant bounded by the two graphs and the x-axis. The line and the curve intersect at point P.



(a) Find the coordinates of P.

intersect

$\frac{2}{3}x = \sqrt{1 - \frac{x^2}{4}}$

$x = 1.2$

$y(1.2) = \frac{2}{3}(1.2)$
$= 0.8$

So, P is at $(1.2, 0.8) = \left(\frac{6}{5}, \frac{4}{5}\right)$

(b) Set up and evaluate an integral expression with respect to x that gives the area of S.

$$\text{Area} = \int_0^{6/5} \left(\frac{2}{3}x - 0\right)dx + \int_{6/5}^{2} \left(\sqrt{1 - \frac{x^2}{4}} - 0\right)dx$$
$$= 0.927$$

(b) Find a polar equation to represent curve C.

$y = \sqrt{1 - \frac{x^2}{4}}$

$r\sin\theta = \sqrt{1 - \frac{(r\cos\theta)^2}{4}}$

$r^2\sin^2\theta = 1 - \frac{1}{4}r^2\cos^2\theta$

$r^2\sin^2\theta + \frac{1}{4}r^2\cos^2\theta = 1$

$r^2\left(\sin^2\theta + \frac{1}{4}\cos^2\theta\right) = 1$

$r^2 = \frac{1}{\sin^2\theta + \frac{1}{4}\cos^2\theta} \cdot \frac{4}{4}$

$r^2 = \frac{4}{4\sin^2\theta + \cos^2\theta}$

(d) Use the polar equation found in (c) to set up and evaluate an integral expression with respect to the polar angle θ that gives the area of S.

$y = \frac{2}{3}x$

$r\sin\theta = \frac{2}{3}r\cos\theta$

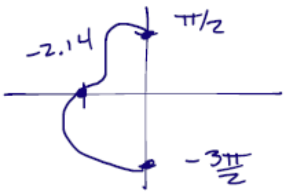
$\tan\theta = \frac{2}{3}$

$\theta = \tan^{-1}\left(\frac{2}{3}\right)$

$$\text{Area} = \frac{1}{2}\int_0^{\arctan\left(\frac{2}{3}\right)} \frac{4}{4\sin^2\theta + \cos^2\theta}\,d\theta$$
$$= 0.927$$

17. A curve is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \cos(3\theta)$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, where r is measured in meters and θ is measured in radians.

(a) Find the area bounded by the curve and the y-axis.



$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{2}}^{3\pi/2} (\theta + \cos 3\theta)^2 \, d\theta$$

$$= 19.674 \text{ or } 19.675$$

(b) Find the angle θ that corresponds to the point on the curve with y-coordinate -1.

$y = -1$

$r \sin\theta = -1$

$(\theta + \cos 3\theta)\sin\theta = -1$

$\underbrace{(\theta + \cos 3\theta)\sin\theta + 1}_{y1 \text{ (function mode)}} = \underbrace{0}_{y2}$

$\theta = 3.484$ or 3.485

(c) For what values of θ, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ is $\frac{dr}{d\theta}$ positive? What does this say about r?

$r = \theta + \cos 3\theta$

$\frac{dr}{d\theta} = \underbrace{1 - 3\sin 3\theta}_{y1} > \underbrace{0}_{y2 \text{ (function mode)}}$

$r \in (\frac{\pi}{2}, \underset{2.208}{2.207}) \cup (3.028, 4.302)$

On these intervals, the graph of $r(\theta)$ is moving away from the pole/origin.

(d) Find the value of θ on the interval $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ that corresponds to the point on the curve with the greatest distance from the origin. What is this greatest distance? Justify your answer.

Maximize r

$\frac{dr}{d\theta} = 0$

$r = 2.207 = A$ (store as A)

$r = 3.028 = B$

$r = 4.302 = C$

Justification

$r(\frac{\pi}{2}) = 1.570$

$r(A) = 3.150$

$r(B) = 2.085$

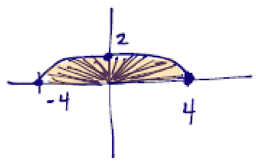
$r(C) = 5.244 \leftarrow$ MAX

$r(\frac{3\pi}{2}) = 4.712$

So, graph is furthest from pole/origin at $\theta = 4.302$ radians. At this angle, the graph is 5.244 units from the pole/origin.

18. A region R in the xy-plane is bounded below by the x-axis and above by the polar curve defined by $r_1 = r = \dfrac{4}{1+\sin\theta}$ for $0 \le \theta \le \pi$.

(a) Find the area of R by evaluating an integral in polar coordinates.



$$\text{Area} = \frac{1}{2}\int_0^{\pi}\left(\frac{4}{1+\sin\theta}\right)^2 d\theta$$

or

$$= \frac{1}{2}\int_0^{\pi} r_1^2\, d\theta$$

$$= 10.666 \text{ or } 10.667 \text{ or } \frac{32}{3}$$

(b) The curve resembles an arch of the parabola $8y = 16 - x^2$. Convert the polar equation to rectangular coordinates, and prove that the curves are the same.

$$r = \frac{4}{1+\sin\theta}$$

$$r = \frac{4}{1+\frac{y}{r}}\cdot\frac{r}{r}$$

$$r = \frac{4r}{r+y}$$

$$1 = \frac{4}{r+y}$$

$$r+y = 4$$

$$r = 4-y$$

$$\sqrt{x^2+y^2} = 4-y$$

$$x^2+y^2 = 16-8y+y^2$$

$$\boxed{8y = 16-x^2}$$

$$y = 2 - \frac{1}{8}x^2$$

(c) Set up an integral in rectangular coordinates that gives the area of R.

using symmetry

$$\text{Area} = 2\int_0^4\left(2-\frac{1}{8}x^2\right)dx$$

or

without symmetry

$$\text{Area} = \int_{-4}^{4}\left(2-\frac{1}{8}x^2\right)dx$$

x-int

$$2 - \frac{1}{8}x^2 = 0$$

$$2 = \frac{1}{8}x^2$$

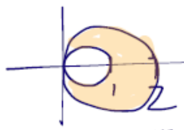
$$16 = x^2$$

$$x = \pm 4$$

Multiple Choice

A 19. Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

(A) $3\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$ (B) $3\int_0^{\pi} \cos^2\theta\, d\theta$ (C) $\frac{3}{2}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$ (D) $3\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$ (E) $3\int_0^{\pi} \cos\theta\, d\theta$



$[0,\pi]$

$$\text{Area} = \frac{1}{2}\int_0^{\pi}\left[(2\cos\theta)^2 - (\cos\theta)^2\right] d\theta$$
$$= \frac{1}{2}\int_0^{\pi}\left(4\cos^2\theta - \cos^2\theta\right) d\theta$$
$$= \frac{3}{2}\int_0^{\pi} \cos^2\theta\, d\theta \quad \text{(Not there!)}$$

-or- using symmetry

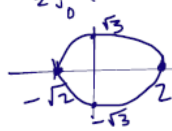
$$2\left[\frac{3}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta\right]$$
$$3\int_0^{\pi/2} \cos^2\theta\, d\theta$$

D 20. The area of the region enclosed by the polar graph of $r = \sqrt{3+\cos\theta}$ is given by which integral?

(A) $\int_0^{2\pi} \sqrt{3+\cos\theta}\, d\theta$ (B) $\int_0^{\pi} \sqrt{3+\cos\theta}\, d\theta$ (C) $2\int_0^{\pi/2} (3+\cos\theta)\, d\theta$

(D) $\int_0^{\pi} (3+\cos\theta)\, d\theta$ (E) $\int_0^{\pi/2} \sqrt{3+\cos\theta}\, d\theta$

$$\text{Area} = \frac{1}{2}\int_0^{2\pi}\left(\sqrt{3+\cos\theta}\right)^2 d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(3+\cos\theta)\, d\theta \quad \text{(not there)}$$



$[0, 2\pi]$

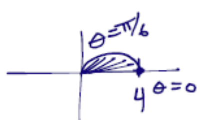
-or- using x-axis symmetry

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi}(3+\cos\theta)\, d\theta\right]$$
$$= \int_0^{\pi}(3+\cos\theta)\, d\theta$$

E 21. The area enclosed by one petal of the 3-petaled rose curve $r = 4\cos(3\theta)$ is given by which integral?

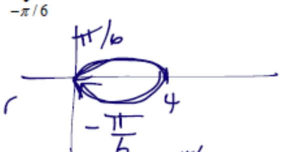
(A) $16\int_{-\pi/3}^{\pi/3} \cos(3\theta)\, d\theta$ (B) $8\int_{-\pi/6}^{\pi/6} \cos(3\theta)\, d\theta$ (C) $8\int_{-\pi/3}^{\pi/3} \cos^2(3\theta)\, d\theta$

(D) $16\int_{-\pi/6}^{\pi/6} \cos(3\theta)\, d\theta$ (E) $8\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$



$r=0$
$4\cos(3\theta)=0$
$\cos 3\theta = 0$
$3\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{6}$

or



$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/6}(4\cos 3\theta)^2 d\theta\right]$$
$$= 16\int_0^{\pi/6}\cos^2 3\theta\, d\theta \quad \text{(not there!)}$$

$$\text{Area} = \frac{1}{2}\int_{-\pi/6}^{\pi/6}(4\cos 3\theta)^2 d\theta$$
$$= 8\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\, d\theta$$