Math 116 — Practice for Exam 3

Generated November 18, 2018

Name: SOLUTIONS

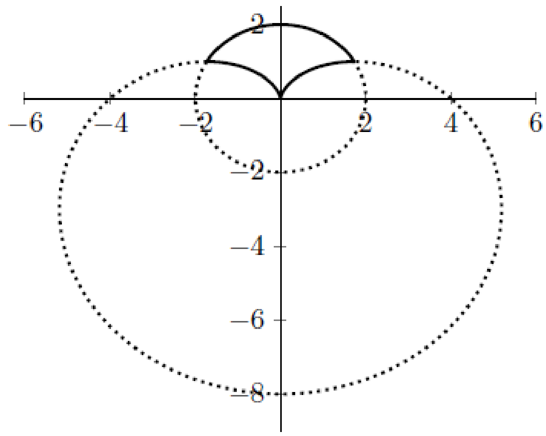
Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2014	3	7	alpaca pool	8	
Winter 2016	3	9	can opener	6	
Winter 2015	3	10	ladybugs2	12	
Winter 2012	2	2	spiral	11	
Winter 2013	2	8	peanut	14	
Winter 2014	2	9	pond rock	10	
Fall 2017	2	4	2p1 orbital	13	
Fall 2012	2	5		12	
Total				86	

Recommended time (based on points): 86 minutes

7. [8 points] Roy the alpaca is designing a pool and a deck for his family. The pool has the shape of a cardioid whose equation is given by $r = 4 - 4\sin(\theta)$ where r is in meters and θ is a number between 0 and 2π. The deck will be built in the region that lies inside the circle $x^2 + y^2 = 4$ and outside the cardioid. The deck is depicted in the figure as the region enclosed by the solid lines



a. [1 point] Write the equation for the circle $x^2 + y^2 = 4$ in polar coordinates.

Solution: $r^2 = 4$ so $r = 2$

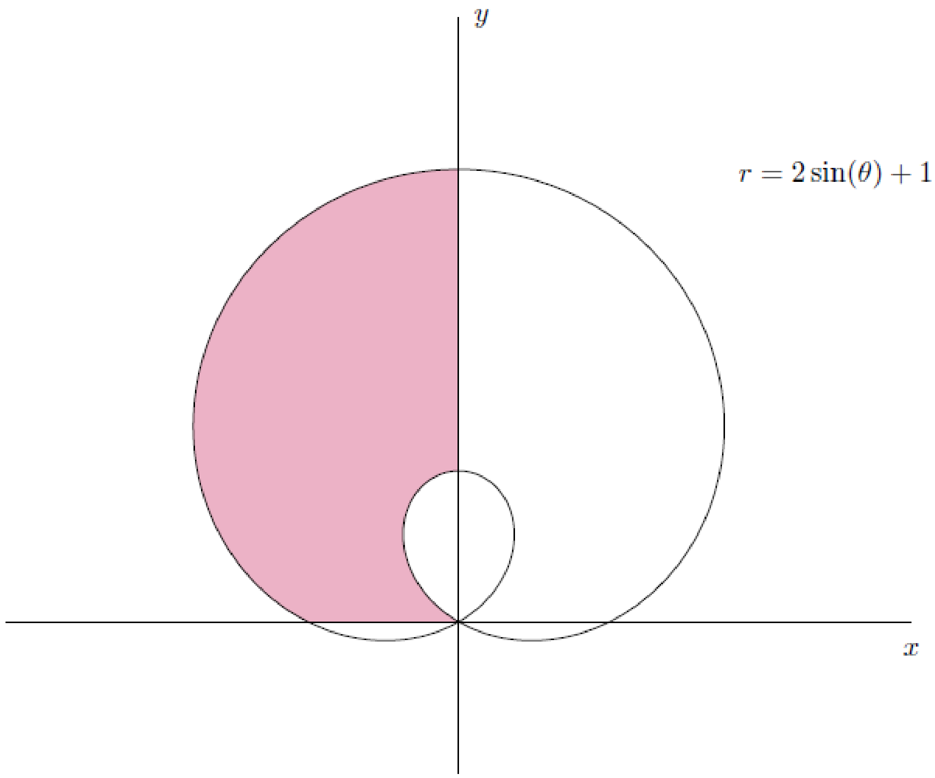
b. [2 points] Find the values of θ between 0 and 2π where the cardioid and the circle intersect.

Solution: Setting the two equations equal to each other we have $2 = 4 - 4\sin(\theta)$ thus $\sin(\theta) = \frac{1}{2}$. Therefore $\theta = \pi/6, 5\pi/6$.

c. [5 points] Write an expression involving integrals that gives the area of the region where the deck will be built. Do not evaluate your expression.

Solution: $\int_{\pi/6}^{5\pi/6} 2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\, d\theta = 4\pi/3 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\, d\theta$.

9. [6 points] O-guk is creating a can opener to open his many cans of juice. The opener is in the shape of the shaded region enclosed by the two loops of the polar curve $r = 2\sin(\theta) + 1$ and the x- and y-axes.

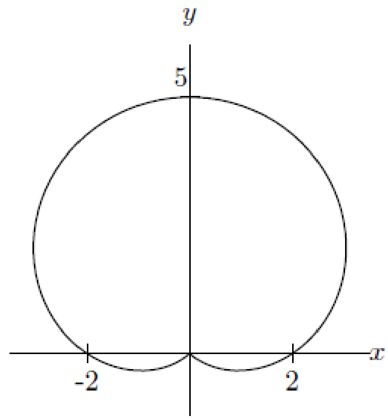


Write an expression involving integrals that gives the total area of the shaded region.

Solution:

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2\sin\theta + 1)^2\, d\theta - \frac{1}{2}\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}} (2\sin\theta + 1)^2\, d\theta$$

10. [12 points] Vic is watching the ladybugs run around in his garden. His garden is in the shape of the outer loop of a cardioid with polar equation $r = 2 + 3\sin\theta$ where r is measured in meters and θ is measured in radians. The outline of the garden is pictured below for your reference. At a time t minutes after he begins watching, Apple, his favorite red ladybug, is at the xy-coordinate $(\sin^2 t, \cos^2 t)$, and Emerald, his prized green ladybug, is at the xy-coordinate $(-\cos(2t), \sin(2t) + 1.5)$. Vic watches the ladybugs for 2π minutes.



Using the information above, circle the correct answer for each part below. There is only one correct answer for each part. You do not need to show your work.

a. [3 points] Which of the following integrals gives the area of the garden?

Solution:

A) $\frac{1}{2}\int_0^{\pi} (2 + 3\sin\theta)^2 d\theta$

B) $\frac{1}{2}\int_{\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta$

C) $\frac{1}{2}\int_{-\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta$

D) $\frac{1}{2}\int_0^{2\pi} (2 + 3\sin\theta)^2 d\theta$

E) $\frac{1}{2}\int_{2\pi-\arcsin(\frac{2}{3})}^{4\pi-\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta$

b. [3 points] Which of the following is **not** true about Apple while Vic is watching?

Solution:

A) Apple runs through the point $(\frac{1}{2}, \frac{1}{2})$ more than once.

B) Apple crosses the path made by Emerald exactly 4 times.

C) Apple's speed is zero at least once.

D) Apple does not leave the garden.

E) Apple is moving faster than Emerald for some of the time.

c. [3 points] How far does Emerald run while Vic is watching?

Solution:

A) π meters

B) 2π meters

C) 4π meters

D) $\sqrt{8}\pi$ meters

E) 8π meters

d. [3 points] After the 2π minutes, Vic stops watching. Apple runs from the point $(x, y) = (0, 1)$ in the positive y-direction with speed of $g(T) = 5Te^{-T}$, T seconds after Vic stops watching. Which of the following is true?

Solution:

A) Apple leaves the garden eventually, but never runs further than 5 meters total.

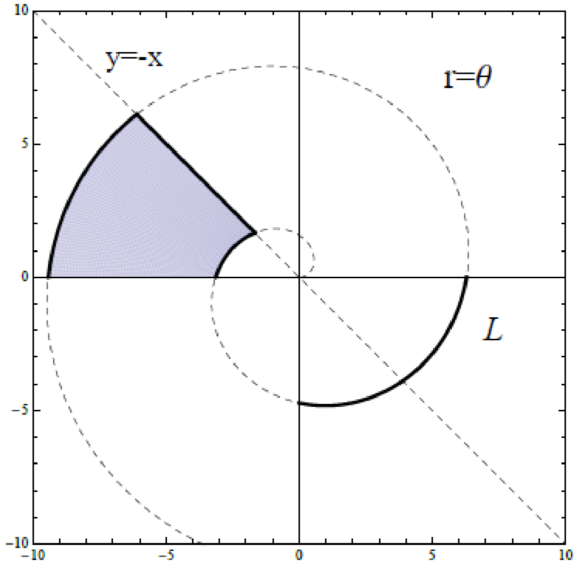
B) Apple's speed is always increasing after Vic stops watching.

C) If given enough time, Apple would eventually be more than 1000 meters from the garden.

D) Apple is still in the garden 5 minutes after Vic stops watching.

E) Apple changes direction, eventually.

2. [11 points] Consider the graph of the spiral $r = \theta$ for $\theta \geq 0$.



In the following questions, write an expression involving definite integrals that computes the values of the following quantities (you do not need to evaluate any integrals) :

a. [4 points] The length of the arc L.

Solution: The arc length of a polar curve is given by the formula

$$\int_{\theta_1}^{\theta_2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} d\theta.$$

Therefore,

$$\text{Length of } L = \int_{3\pi/2}^{2\pi} \sqrt{\theta^2 + 1}\, d\theta.$$

b. [7 points] The area of the shaded region.

Solution: The of the region inside of a polar curve is

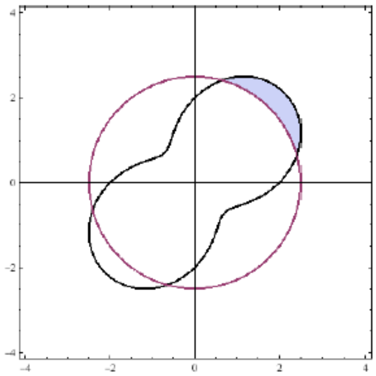
$$\frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 d\theta.$$

We have to take the outer area minus the inner area. We can write the line $y = -x$ as $\theta = \frac{3\pi}{4} + 2\pi k$ for some k. The outer curve is on the second pass around the origin, and the inner curve is on the first time around the origin. Wo we get

$$A = \frac{1}{2}\int_{11\pi/4}^{3\pi} \theta^2\, d\theta - \frac{1}{2}\int_{3\pi/4}^{\pi} \theta^2\, d\theta.$$

8. [14 points]

a. [6 points] Find a definite integral that computes the shaded area outside the circle $r = \frac{5}{2}$ and inside the curve given by $r = 2 + \sin 2\theta$ in the graph below.



Solution:

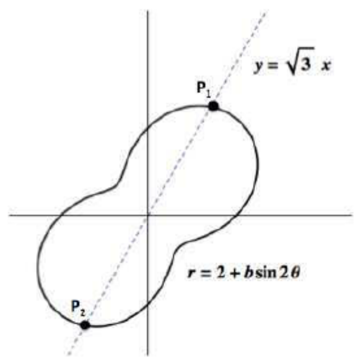
$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((2+\sin(2\theta))^2 - \left(\frac{5}{2}\right)^2\right) d\theta.$$

Here we found the limits of integration by solving for where $5/2 = 2 + \sin 2\theta$ for θ in the first quadrant.

b. [4 points] Find the polar coordinates (r, θ) of the points where the line $y = \sqrt{3}\,x$ intersects the graph of $r = 2 + b\sin 2\theta$. Here the constant $0 < b < 2$. Your answers may include b.

P_1=____________________

P_2=____________________



Solution: We want $\tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. In the first case, we have

$$P_1 = (r,\theta) = \left(2 + b\sin\left(\frac{2\pi}{3}\right), \frac{\pi}{3}\right) = \left(2 + \frac{\sqrt{3}}{2}b, \frac{\pi}{3}\right).$$

$$P_2 = (r,\theta) = \left(2 + b\sin\left(\frac{8\pi}{3}\right), \frac{4\pi}{3}\right) = \left(2 + \frac{\sqrt{3}}{2}b, \frac{4\pi}{3}\right).$$

c. [4 points]

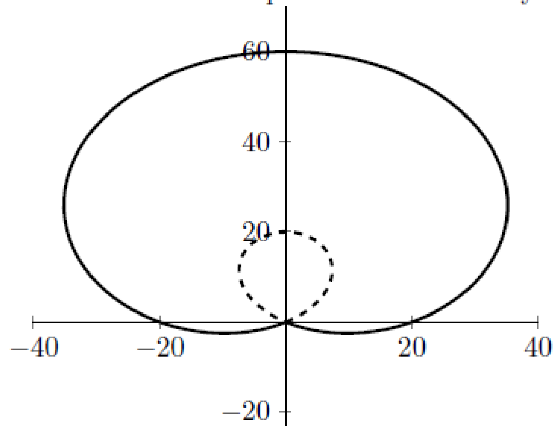
i) Find the equation in polar coordinates of the line $x = 0$.

Solution: $\theta = \frac{\pi}{2}$.

ii) Find the equation in polar coordinates of the line $y = 4$.

Solution: $y = r\sin\theta = 4$, so $r = \frac{4}{\sin\theta}$.

9. [10 points] Linda is designing a pond with a flat rock at one end. The rock plus the pond are in the shape of a cardioid. Plans for her pond design are depicted below. The cardioid has equation $r = 20 + 40\sin\theta$ where r is in feet and θ is in radians. The inner loop of the cardioid forms the shape of the rock and the outer loop forms the boundary of the pond.



a. [2 points] Find all values of θ between 0 and 2π for which $r = 0$.

Solution: First we set $r = 20 + 40\sin\theta = 0$. Therefore rearranging $\sin\theta = -\frac{1}{2}$. The solutions between 0 and 2π are $\theta = 7\pi/6, 11\pi/6$.

b. [4 points] Write an integral or sum of integrals which give(s) the perimeter of the boundary of the pond. Note this is the perimeter of the part of the cardioid drawn with a solid line.

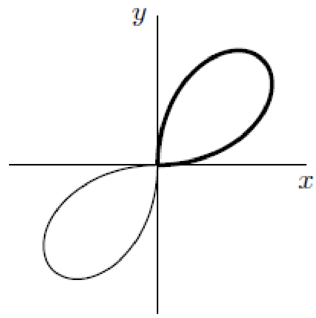
Solution:
We will need to use the polar arc length formula so we need to calculate $r' = 40\cos\theta$. The arc length can be written as a single integral $\int_{-\pi/6}^{7\pi/6} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta$. Writing the arc length as two integrals we get $\int_0^{7\pi/6} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta + \int_{11\pi/6}^{2\pi} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta$

c. [4 points] Write an integral or sum of integrals which give(s) the area of the top of the rock. Note this is the area enclosed by the dashed part of the cardioid.

Solution:
Now we need to use the polar area formula $\int_{7\pi/6}^{11\pi/6} \frac{1}{2}(20 + 40\sin\theta)^2\, d\theta$.

4. [13 points] The orbit of a single electron around the nucleus of an atom is determined by the energy level of that electron and by the other electrons orbiting the nucleus. We can model one electron's orbital in two-dimensions as follows. Suppose that the nucleus of an atom is centered at the origin. Then the (so-called "$2p_1$") orbital has the shape shown below.

This shape is made up of two regions that we call "lobes". The outer edge of the lobes are described by the polar equation $r = k\sin(2\theta)$ for some positive constant k. Note that only the relevant portion of the polar curve $r = k\sin(2\theta)$ is shown.



The "top lobe" is the portion in the first quadrant (shown in bold).

a. [2 points] For what values of θ with $0 \le \theta \le 2\pi$ does the polar curve $r = k\sin(2\theta)$ pass through the origin?

Solution: The curve passes through the origin when $r = 0$, i.e. when $\sin(2\theta) = 0$, or $\boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 2\pi.}$

b. [3 points] For what values of θ does the polar curve $r = k\sin(2\theta)$ trace out the "top lobe"? Give your answer as an interval of θ values.

Solution: From part (a), we are at the origin when $\theta = 0$; as θ increases from 0, the radius $k\sin(2\theta)$ increases and then decreases (since k is positive). So we finish the lobe when we get to the second value of θ at which the curve intersects the origin, $\theta = \frac{\pi}{2}$. The final answer is thus $\boxed{0 \le \theta \le \pi/2 \text{ or, equivalently, the closed interval } [0, \pi/2]}$.

c. [4 points] Write, but do **not** evaluate, an integral that gives the area of the top lobe.

Solution: Using the formula for polar area and the values of θ from part ,

$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(f(\theta))^2\, d\theta = \int_0^{\pi/2} \frac{(k\sin(2\theta))^2}{2}\, d\theta.$$

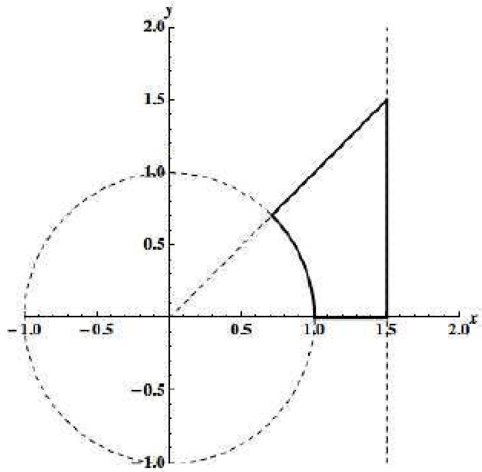
d. [4 points] Imagine that an electron lies within the top lobe of this orbital, but is as far away from the origin as possible. What are the polar coordinates of this point of greatest distance from the origin? Your answer may involve the constant k.

Solution: Maximizing distance from the origin means maximizing $|r|$, so want $\sin(2\theta) = \pm 1$. For θ in the interval $0 \le \theta \le \pi/2$, this implies that $2\theta = \pi/2$ so $\theta = \pi/4$. When $\theta = \pi/4$, we have $r = k\sin(\pi/2) = k$.

Therefore, the polar coordinates for this point are $\boxed{(r, \theta) = \left(k, \frac{\pi}{4}\right)}$.

5. [12 points] Solve each of the following problems.

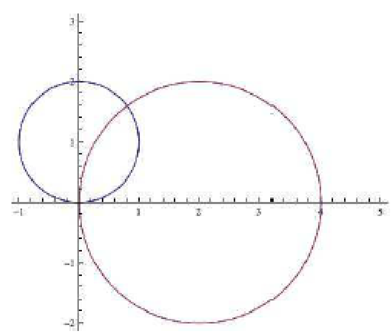
a. [4 points] Give inequalities for r and θ that describe the region shown below in polar coordinates. The region is bounded by the circle $x^2 + y^2 = 1$, the line $y = x$, the x-axis and the vertical line $x = 1.5$.



Solution: The line $y = x$ is the polar line $\theta = \frac{\pi}{4}$, so the limits for θ are $0 \le \theta \le \frac{\pi}{4}$. The values of r range from $r = 1$ on the circle to the line $r\cos\theta = 1.5$, or $r = \frac{1.5}{\cos\theta}$. So the region is

$$\begin{cases} 0 \le \theta \le \frac{\pi}{4} \\ 1 \le r \le \frac{1.5}{\cos\theta} \end{cases}$$

b. [8 points] The functions in polar coordinates $r = 2\sin\theta$ and $r = 4\cos\theta$ represent the circles shown below



Let A be the area of the intersection of these circles. Find an expression involving definite integrals in polar coordinates that computes the value of A. You do not need to evaluate the integrals.

Solution: The curves intersect where $2\sin\theta = 4\cos\theta$, or $\tan\theta = 2$. On the interval $0 \le \theta \le \arctan(2)$, $r = 2\sin\theta$ is the outside curve. On the interval $\arctan(2) \le \theta \le \frac{\pi}{2}$, $r = 4\cos\theta$ is the outside curve. The area is

$$A = \frac{1}{2}\int_0^{\arctan(2)} 4\sin^2\theta d\theta + \frac{1}{2}\int_{\arctan(2)}^{\frac{\pi}{2}} 16\cos^2\theta d\theta.$$