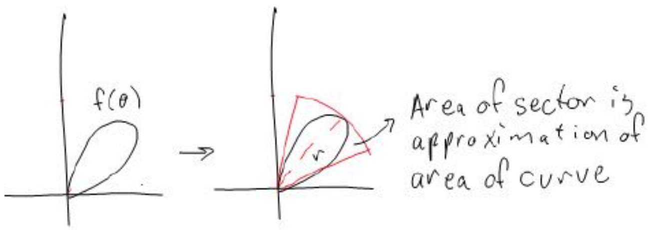
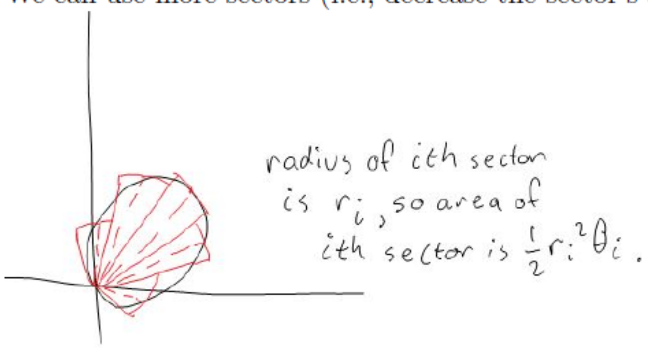
Areas of Polar Curves

In this section we will find a formula for determining the area of regions bounded by polar curves. To do this, wee again make use of the idea of approximating a region with a shape whose area we can find, then use calculus to make the approximations exact.

We can approximate the area of the polar curve below using a sector of a circle:



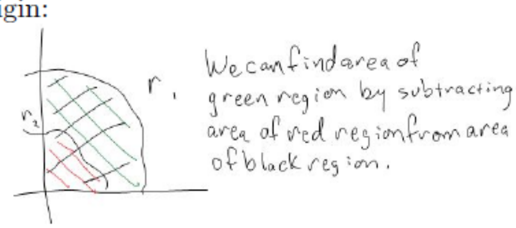
We can use more sectors (i.e., decrease the sector's angle θ) to get a better approximation:



The area of a sector of a circle of radius r is given by $A = \frac{1}{2}\theta r^2$, where θ is the central angle of the sector; if the region whose area we wish to find is bounded between $\theta = \alpha$ and $\theta = \beta$, then the area between the curve and the origin is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

We can use a similar idea to find the area between two polar curves; for instance, to determine the area between r_1 and r_2 in the picture below (the green shaded area), find the area bounded between the outer curve and the origin, then subtract the area between the inner curve and the origin:



So the area of the region bounded between $r_1(\theta)$ and $r_2(\theta)$, where $r_1(\theta) \geq r_2(\theta)$ for $\alpha \leq \theta \leq \beta$

is given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r_1^2 - r_2^2 d\theta.$$

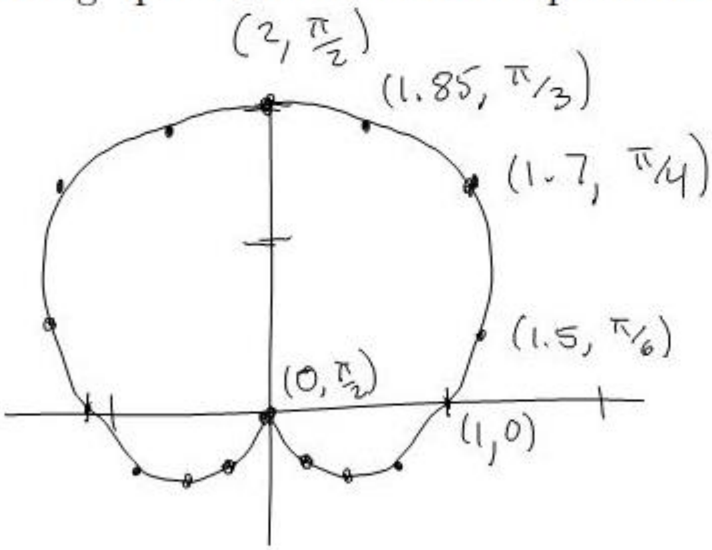
To determine the area of a region described by a polar equation, follow these steps:

1. Graph the region
2. If necessary, determine where the curves intersect
3. Set up an integral for each region with distinct outer or inner boundaries
4. Determine the bounds of integration for each region
5. Evaluate the integrals.

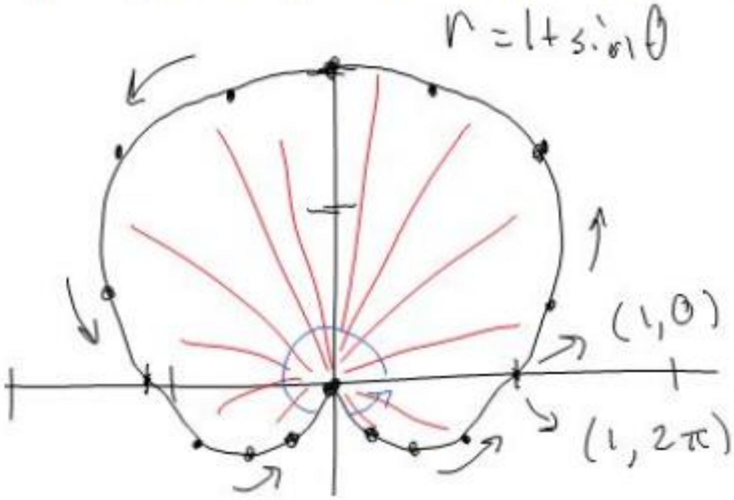
Examples

Find the area of the cardioid $r = 1 + \sin\theta$.

We graphed this curve in the previous section:



The radius of the curve is always $r = 1 + \sin\theta$:

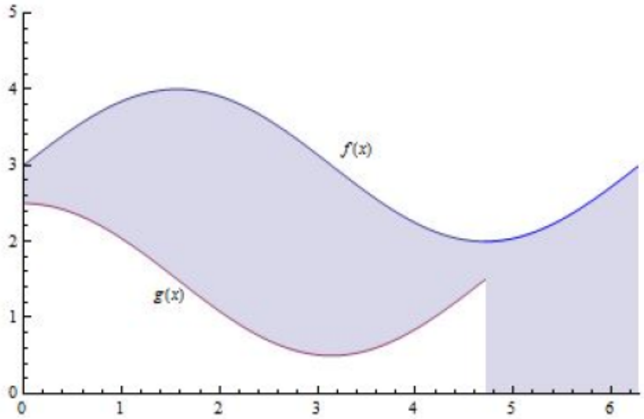


In addition, the curve extends from $\theta = 0$ to $\theta = 2\pi$, so the area is given by

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \sin^2\theta d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \frac{1}{2}(1-\cos(2\theta))d\theta \\
&= \frac{1}{2}(\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin(2\theta))\Big|_0^{2\pi} \\
&= \frac{1}{2}(\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin(2\theta))\Big|_0^{2\pi} \\
&= \frac{1}{2}(3\pi - 2 + 2) \\
&= \frac{3\pi}{2}.
\end{aligned}$$

Finding areas in polar coordinates can be a bit complicated because points along the curve are being measured with respect to the origin in terms of r and θ, *not* with respect to the x axis in terms of x and y. However there is a nice analogue between the two types of area calculations.

Consider the curves $f(x)$ and $g(x)$ below, graphed in Cartesian coordinates in the xy plane:

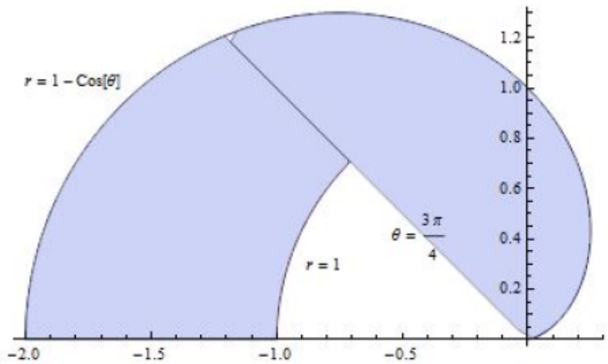


To find the area of the shaded region, we will need to set up two different integrals. Notice that points on $f(x)$ from $x = 0$ to $x = 3\pi/2$ cannot "see" the x axis because they are blocked by $g(x)$; so to find the area of this region, we must subtract $g(x)$ from $f(x)$. On the other hand, points on $f(x)$ between $x = 3\pi/2$ and $x = 2\pi$ can see the x axis since $g(x)$ is no longer in the way, so we do not need to subtract anything from $f(x)$ on this region. Thus to find the area of the shaded region above, we need two separate integrals: the area is given by

$$A = \int_0^{3\pi/2} f(x) - g(x)dx + \int_{3\pi/2}^{2\pi} f(x)dx.$$

Let's think about the analogue for polar curves in the xy plane. Keep in mind that points on polar curves are measured with respect to the *origin*, not the x axis, and the area enclosed by a polar curve is enclosed between the curve and the origin.

Consider the curves $r = 1 - \cos\theta$, $0 \leq \theta \leq 2\pi$ and $r = 1$, $3\pi/4 \leq \theta \leq 2\pi$:

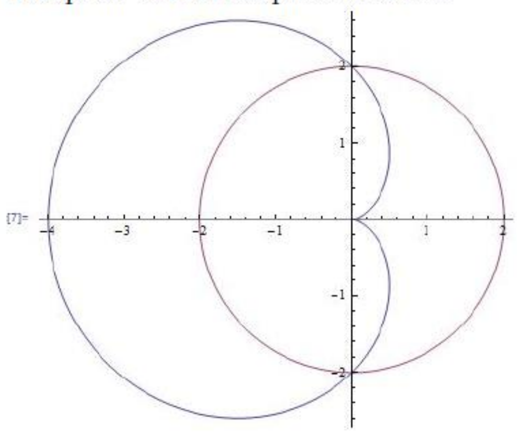


To find the area of the shaded region, we again need two separate integrals. Points on $r = 1 - \cos\theta$ cannot "see" the origin between $\theta = 3\pi/4$ and $\theta = 2\pi$ because they are blocked by $r = 1$. So to find the area of this region, we need to subtract $r = 1$ from $r = 1 - \cos\theta$. On the other hand, points on $r = 1 - \cos\theta$ between $\theta = 0$ and $\theta = 3\pi/4$ are not blocked from viewing the origin, so we do not need to subtract anything from $r = 1 - \cos\theta$ on this region. So the area shaded above is given by

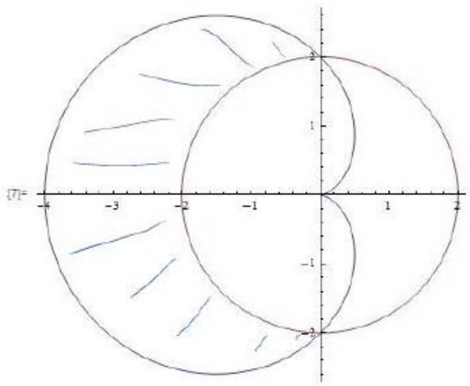
$$A = \int_0^{3\pi/4} (1 - \cos\theta)^2 d\theta + \int_{3\pi/4}^{2\pi} (1 - \cos\theta)^2 - 1 d\theta.$$

Find the area of the region in inside of $r = 2 - 2\cos\theta$ and outside of $r = 2$.

The polar curves are plotted below:



The region whose are we wish to find is shaded below:

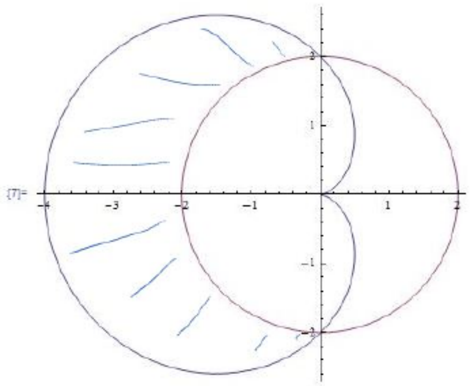


We need to determine where the two curves intersect, i.e. where $2 - 2\cos\theta = 2$.

$$\begin{aligned} 2 - 2\cos\theta = 2 \ &\Rightarrow \ \cos\theta = 0 \\ &\Rightarrow \ \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}. \end{aligned}$$

So the first point of intersection in the graph is $\frac{\pi}{2}$, and the second is $\frac{3\pi}{2}$.
The regions area can be found by determining the area enclosed by $2 - 2\cos\theta = 2$ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and subtracting off the area enclosed by $r = 2$ on the same bounds. So the area is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 - 2\cos\theta)^2 - (2)^2 \ d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta - 4 \ d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos^2\theta - 4\cos\theta \ d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 + \cos(2\theta) - 4\cos\theta \ d\theta \\ &= \theta + \frac{1}{2}\sin(2\theta) - 4\sin\theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \frac{3\pi}{2} + 4 - \frac{\pi}{2} + 4 \\ &= \pi + 8. \end{aligned}$$



We need to determine where the two curves intersect, i.e. where $2 - 2\cos\theta = 2$.

$$\begin{aligned} 2 - 2\cos\theta = 2 \quad &\Rightarrow \quad \cos\theta = 0 \\ &\Rightarrow \quad \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}. \end{aligned}$$

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$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 - 2\cos\theta)^2 - (2)^2 \, d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta - 4 \, d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos^2\theta - 4\cos\theta \, d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 + \cos(2\theta) - 4\cos\theta \, d\theta \\ &= \theta + \frac{1}{2}\sin(2\theta) - 4\sin\theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \frac{3\pi}{2} + 4 - \frac{\pi}{2} + 4 \\ &= \pi + 8. \end{aligned}$$

Exercise:
Set up an integral for the area of the region enclosed *between* the two curves from the previous example.