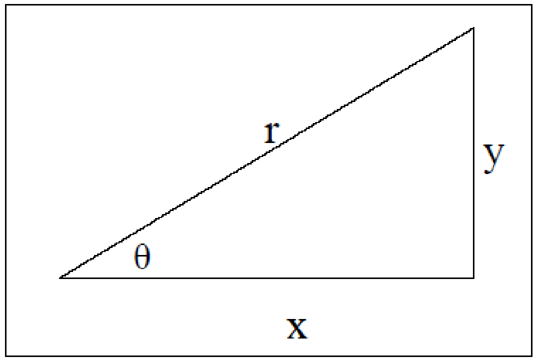
CALCULUS BC NOTES
POLAR

Know the equations ALL based on the following graph:



SO:

$$r^2 = x^2 + y^2 \Rightarrow r = \pm\sqrt{x^2+y^2} \Rightarrow \boxed{r = \sqrt{x^2+y^2}} \quad \text{[usually don't need the negative value]}$$

$$\boxed{\tan\theta = \frac{y}{x}} \quad \text{AND} \quad \sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta} \quad \text{AND} \quad \cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$$

Make sure that you can convert from polar (r,θ) to rectangular (x,y) and vice–versa.

<u>AREA IN POLAR</u>

The area of a sector is: Area $=\frac{1}{2}r^2\theta$.

Concept: We will add together an infinite number of infinitely thin $(d\theta)$ sectors to find the exact area under the polar curve.

So, area inside a polar curve is given by: $\boxed{\text{Area} = \frac{1}{2}\int_{\theta=}^{\theta=} r^2\, d\theta}$ AND

The area BETWEEN polar curves {Concept similar to Washers} is given by:

$$\boxed{\text{Area} = \frac{1}{2}\int_{\theta=}^{\theta=} \left(R^2 - r^2\right) d\theta}$$

CALCULUS BC NOTES

Example #1

Find the area inside $r = 2 + 2\cos\theta$

Finding the limits of integration can be the "tricky" part.

Method #1:

Use your calculator to TRACE.

Scroll to the right and watch the values of θ.

One revolution occurs when θ goes from $\boxed{0 \text{ to } 2\pi}$

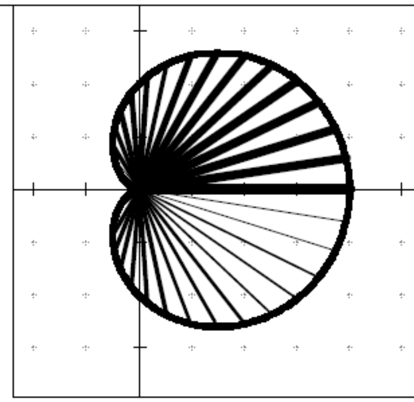
Method #2:

Set $r = 0$ and solve for values of θ.

$r = 2 + 2\cos\theta = 0 \Rightarrow 2\cos\theta = -2 \Rightarrow \cos\theta = -1$

$\boxed{\theta = \pi \text{ or } 3\pi}$

One revolution occurs when θ goes from $\boxed{\pi \text{ to } 3\pi}$



Solution:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (2+2\cos\theta)^2 d\theta \;\Rightarrow\; \frac{1}{2}\int_0^{2\pi} 4\left(1+2\cos\theta+\boxed{\cos^2\theta}\right)d\theta \;\Rightarrow\; 2\int_0^{2\pi}\left(1+2\cos\theta+\boxed{\frac{1}{2}\cos 2\theta+\frac{1}{2}}\right)d\theta \;\Rightarrow$$

$$2\int_0^{2\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta \;\Rightarrow\; 2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 2\left[\left(\frac{3}{2}(2\pi)+0+0\right)-(0+0+0)\right] = \boxed{\boxed{6\pi}}$$

Method #2: $$2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\pi}^{3\pi} = 2\left[\left(\frac{3}{2}(3\pi)+0+0\right)-\left(\frac{3}{2}\pi+0+0\right)\right] = 2\left[\frac{9}{2}\pi-\frac{3}{2}\pi\right] = \boxed{\boxed{6\pi}}$$

CALCULUS BC NOTES

Finding the limits of integration can be the "tricky" part.
There are many methods for doing this. Here are some of them:

Method #1:

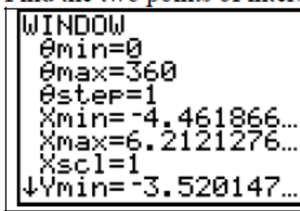
Use your calculator to TRACE.
Find the two points of intersection between the two equations.

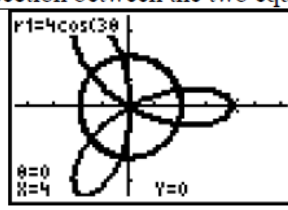
Tracing the darkest area yields 0 and 0.35 which equals $\boxed{0 \text{ and } \dfrac{\pi}{9}}$

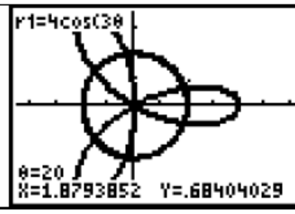
The downfall of this method is that you get a decimal and it's hard to convert that to the exact value $\left(\theta = \dfrac{\pi}{9}\right)$.

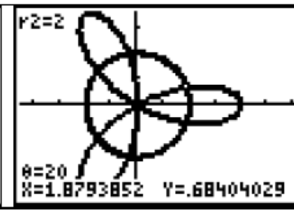
Method #2:

PUT CALCULATOR IN DEGREE MODE and then use your calculator to TRACE
Find the two points of intersection between the two equations.









Window: make sure to change θ step to one DEGREE
Look at upper left hand corner of the last two graphs.
Notice that when I jump from r1 to r2 (up arrow) the x and y values are identical.
This verifies that this is an **intersection**. **THIS IS IMPORTANT.**
The downfall of this method is that you have to go back to RADIAN MODE in order to integrate.
Also when you go back to RADIAN MODE check your window. $[\theta\text{min} = 0,\ \theta\text{max} = 2\pi,\ \theta\text{step} = 0.1]$

Method #3:

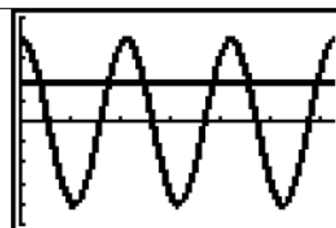
Set the equations equal to each other to find the points of intersection.

Notice that the polar graph intersect SIX times from 0 to 2π.

$4\cos 3\theta = 2 \Rightarrow \cos 3\theta = \dfrac{1}{2} \Rightarrow 3\theta = \cos^{-1}\dfrac{1}{2} \Rightarrow$

$3\theta = \dfrac{\pi}{3} + 2\pi n$ AND $3\theta = \dfrac{5\pi}{3} + 2\pi n \Rightarrow$

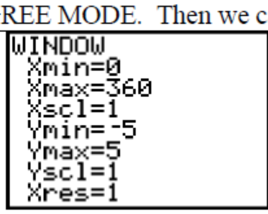
$\theta = \dfrac{\pi}{9} + \dfrac{2\pi}{3}n$ or $\theta = \dfrac{5\pi}{9} + \dfrac{2\pi}{3}n \Rightarrow \boxed{\theta = \dfrac{\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}\dfrac{11\pi}{9}\dfrac{13\pi}{9}\dfrac{17\pi}{9}}$

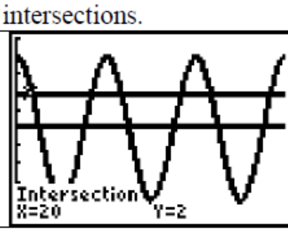


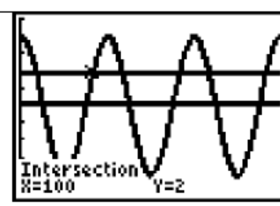
Method #4

Another way of finding the intersection r = 2 and r = 4cos(3θ)

Instead of graphing in polar, we can go to MODE>FUNCTION and graph y = 4cos(3θ) and y = 2 in DEGREE MODE. Then we calculate the intersections.







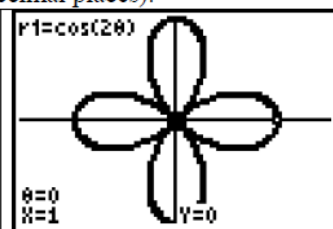
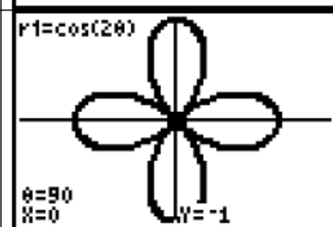
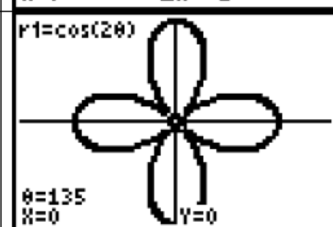
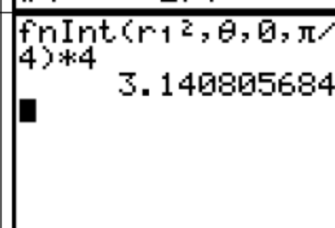
Solution:

$$\text{Area} = 6\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{9}} (4\cos 3\theta)^2 - (2)^2 d\theta \;\Rightarrow\; 3\int_0^{\frac{\pi}{9}} 16\cos^2 3\theta - 4\, d\theta$$

$$3\int_0^{\frac{\pi}{9}} 8 + 8\cos 6\theta - 4\, d\theta \;\Rightarrow\; 3\int_0^{\frac{\pi}{9}} 4 + 8\cos 6\theta\, d\theta \;\Rightarrow 3\left[4\theta + \frac{4}{3}\sin 6\theta\right]_0^{\frac{\pi}{9}} \;\Rightarrow\; 3\left[\frac{4\pi}{9} + \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)\right] = \boxed{\frac{4\pi}{3} + \frac{4\sqrt{3}}{2}}$$

Method 1:

Find the area inside of the four leaved Brose: r = 2cos(2θ) (Round answer to 3 decimal places).

To find the points where θ is 0, graph r = cos(2θ) in degrees.	
At θ = 0 degrees and 90 degrees, r = 1.	
At θ = 45 degrees and 135 degrees, r = 0. Thus we use 45 degrees and 135 degrees as our limits of integration (Remember to multiply by 4 since this is only 1 leaf).	
We try to integrate in degrees mode and get the wrong answer!	
We go to MODE>RADIANS and integrate again and get the right answer!!!!!