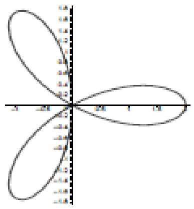
MATH 1020 WORKSHEET 10.4

Area and Arc Length in Polar Coordinates

Area in polar coordinates is found with the formula $A = 1/2 \int_a^b r^2 \, d\theta$
Arc Length has the formula $s = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$
Graphs of polar curves will be given to you on quizzes and exams

Find the area of one petal of the region $r = 2\cos(3\theta)$.
Solution. One must first determine values of θ where $r = 0$.



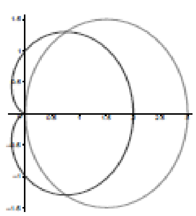
Setting $r = 0$ we find that

$$
\begin{aligned}
0 &= 2\cos(3\theta) \\
&= \cos(3\theta) \\
\text{Thus } 3\theta &= \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}
\end{aligned}
$$

Thus we have that the petal that lies on the x-axis is traced out on the interval $\frac{-\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Note that one can also use symmetry and find the area by calculating $2\times$ the area on the interval $0 \leq \theta \leq \frac{\pi}{6}$. Our area integral becomes

$$
\begin{aligned}
\frac{1}{2}\int_{-\pi/6}^{\pi/6} (2\cos(3\theta))^2 \, d\theta &= 2\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta \\
&= 2\int_{-\pi/6}^{\pi/6} \frac{1}{2}(1+\cos(6\theta))\, d\theta \\
&= \int_{-\pi/6}^{\pi/6} (1+\cos(6\theta))\, d\theta \\
&= \left(\theta + \frac{\sin(6\theta)}{6}\right)\Bigg|_{-\pi/6}^{\pi/6} \\
&= \frac{\pi}{6} - \frac{-\pi}{6} + \frac{\sin(6(\pi/6))}{6} - \frac{\sin(6(-\pi/6))}{6} \\
&= \frac{2\pi}{6} + 0 - 0 \quad = \quad \underline{\underline{\frac{\pi}{3}}}
\end{aligned}
$$

Find the θ values for the points of intersection for the graphs of the equations $r = 1 + \cos\theta$ and $r = 3\cos\theta$.
Solution.



Setting the two equations equal one finds

$$
\begin{aligned}
3\cos\theta &= 1 + \cos\theta \\
2\cos\theta &= 1 \\
\cos\theta &= \frac{1}{2} \\
\text{Thus} \quad \theta &= \underline{\underline{\frac{\pi}{3}}} \quad \text{and} \quad \theta = \underline{\underline{-\frac{\pi}{3}}}.
\end{aligned}
$$

Using the results from the previous problem, find the area inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.

Solution. The area enclosed between the two curves on the interval $\frac{-\pi}{3} \leq \theta \leq \frac{\pi}{3}$ can be found by calculating $2\times$ the area between the two curves on the interval $0 \leq \theta \leq \frac{\pi}{3}$. The calculation follows

$$\begin{aligned}
2 \cdot \left[\frac{1}{2}\int_0^{\frac{\pi}{3}} (3\cos\theta)^2\, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2\, d\theta\right] &= 2\cdot\left(\frac{1}{2}\right)\left[\int_0^{\frac{\pi}{3}} 9\cos^2\theta\, d\theta - \int_0^{\frac{\pi}{3}}\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta\right] \\
&= \int_0^{\frac{\pi}{3}} \left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right) d\theta \\
&= \int_0^{\frac{\pi}{3}} \left(8\cos^2\theta - 1 - 2\cos\theta\right) d\theta \\
&= \int_0^{\frac{\pi}{3}} 8\frac{1}{2}(1 + \cos(2\theta)) - 1 - 2\cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} 4 + 4\cos(2\theta) - 1 - 2\cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{3}} 3 + 4\cos(2\theta) - 2\cos\theta\, d\theta \\
&= \left(3\theta + 2\sin(2\theta) - 2\sin\theta\right)\Big|_0^{\frac{\pi}{3}} \\
&= 3\left(\frac{\pi}{3}\right) + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} - 3\cdot 0 + 2\cdot 0 - 2\cdot 0 \\
&= \pi + 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} \quad = \quad \underline{\underline{\pi}}
\end{aligned}$$

Find the length of $r = 2a\cos\theta$ on the interval $-\pi/2 \leq \theta \leq \pi/2$.

Solution. First we find $\frac{dr}{d\theta}$

$$\frac{dr}{d\theta} = -2a\sin\theta$$

We can now use this result in the polar arc length formula.

$$\begin{aligned}
L &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\, d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2(1)}\, d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2a\, d\theta \\
&= 2a\theta\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \\
&= 2a\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \quad = \quad 2a\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \quad = \quad \underline{\underline{2a\pi}}.
\end{aligned}$$