

INTEGRATION IN POLAR COORDINATES

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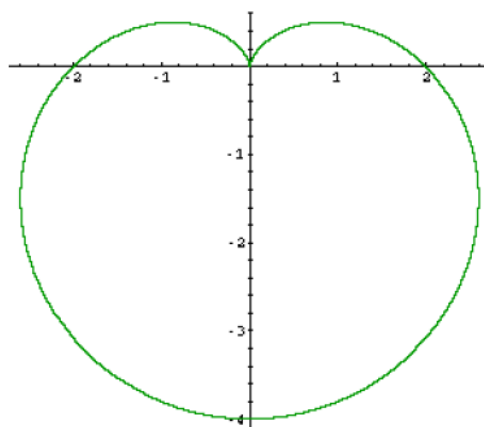
One of the main reasons why we study polar coordinates is to help us to find the area of a region that cannot easily be integrated in terms of x . In this set of notes, I will show how to find the area of the region using polar coordinates. Find the length of the curve using polar coordinates. Find the surface area of the surface of revolution when a polar curve is revolved about an axis.

AREA IN THE PLANE

$$AREA = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

EXAMPLE 1: Find the area bounded by the curve $r = 2 - 2 \sin \theta$.

SOLUTION: Here is the graph of this curve.



This function is symmetric with respect to the y -axis, so I will let $-\pi/2 \leq \theta \leq \pi/2$ and multiply the area by 2.

$$\begin{aligned} A &= 2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (2 - 2 \sin \theta)^2 d\theta \\ &= \int_{-\pi/2}^{\pi/2} (4 - 8 \sin \theta + 4 \sin^2 \theta) d\theta \quad \otimes \\ &\quad \otimes = \int_{-\pi/2}^{\pi/2} (4 - 8 \sin \theta + 2(1 - \cos 2\theta)) d\theta \\ &= \int_{-\pi/2}^{\pi/2} (6 - 8 \sin \theta - 2 \cos 2\theta) d\theta \\ &= 6\theta + 8 \cos \theta - \sin 2\theta \Big|_{-\pi/2}^{\pi/2} = 6\pi \end{aligned} \quad \text{FACT: } \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

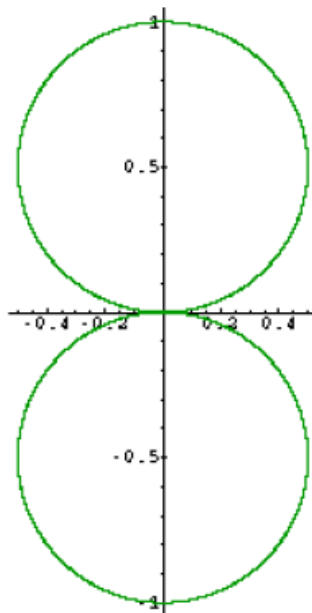
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EXAMPLE 2: Find the area enclosed by the curve $r^2 = a^2 \sin^2 \theta$.

SOLUTION: Here is the graph of this curve.



Since the graph is symmetric with respect to the x-, y-axis and the origin, I will take 4 times the area of a quarter of the graph. I will use the upper right quarter, and for this quarter, θ starts at 0 and ends at $\pi/2$. The equation for the upper circle is $r = a \sin \theta$.

$$\begin{aligned} A &= 4 \int_0^{\pi/2} \frac{1}{2} a^2 \sin^2 \theta d\theta = 2a^2 \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2\theta) d\theta \\ &= a^2 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta = a^2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/2} = \frac{a^2 \pi}{2} \end{aligned}$$

EXAMPLE 3: Find the area of the region bounded by $r = \tan 2\theta$ and the rays $\theta = 0$ and $\theta = \pi/8$.

SOLUTION: The bounds for integration are provided to us in this problem. θ

starts at 0 and ends at $\pi/8$. So let us calculate the area.

$$A = \int_0^{\pi/8} \frac{1}{2} (\tan 2\theta)^2 d\theta = \int_0^{\pi/8} \frac{1}{2} \tan^2 2\theta d\theta \quad \otimes$$

Recall that $\tan^2 2\theta + 1 = \sec^2 2\theta$, so $\tan^2 2\theta = \sec^2 2\theta - 1$.

$$\otimes = \frac{1}{2} \int_0^{\pi/8} (\sec^2 2\theta - 1) d\theta$$

Let $u = 2\theta$, then $du = 2d\theta$.

$$\begin{aligned} &= \frac{1}{2} \left[\frac{1}{2} \int_0^{\pi/4} \sec^2 u du - \int_0^{\pi/8} 1 d\theta \right] \\ &= \frac{1}{2} \left(\frac{1}{2} \tan u \Big|_0^{\pi/4} - \theta \Big|_0^{\pi/8} \right) = \frac{1}{4} - \frac{\pi}{16} \end{aligned}$$

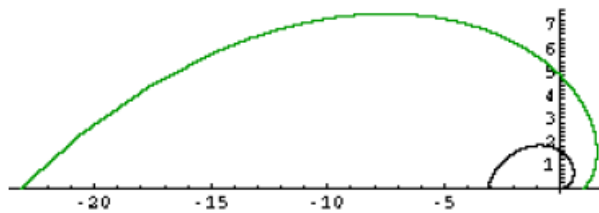
Suppose I needed to find the area of the region enclosed by two polar curves, how could the area formula need to be modified? Do you remember how we found the area between two curves in calculus I? We always wrote the integrand as the outer curve minus the inner curve. We will approach the area of the region enclosed by two polar curves the same way.

AREA OF THE REGION $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta$$

EXAMPLE 4: Find the area of the region bounded by the graphs $r = e^\theta$, $0 \leq \theta \leq \pi$; $r = \theta$, $0 \leq \theta \leq \pi$; the rays $\theta = 0$ and $\theta = \pi$.

SOLUTION: Here is the graph of these curves.



First of all, the bounds for integration are $0 \leq \theta \leq \pi$. The outer curve is $r = e^\theta$, and the inner curve is $r = \theta$. Let us calculate the area.

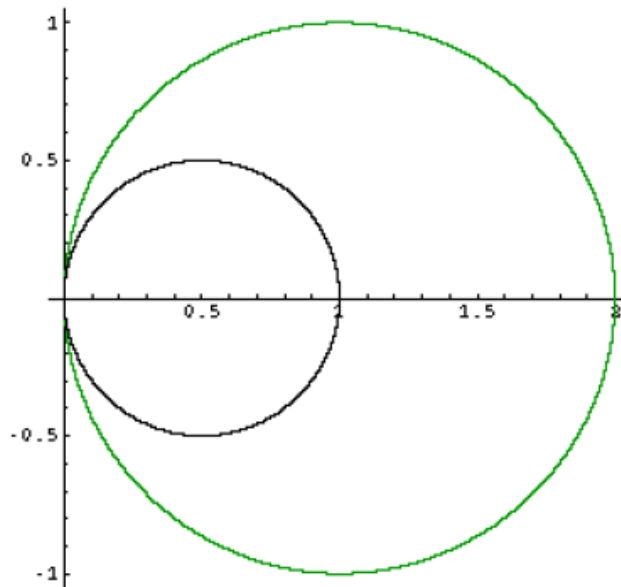
$$A = \int_0^{\pi} \frac{1}{2} (e^{2\theta} - \theta^2) d\theta = \frac{1}{2} \left[\frac{1}{2} \int_0^{2\pi} e^u du - \int_0^{\pi} \theta^2 d\theta \right]$$

Let $u = 2\theta$, then $du = 2 d\theta$.

$$= \frac{1}{2} \left[\frac{1}{2} e^u \Big|_0^{2\pi} - \frac{\theta^3}{3} \Big|_0^{\pi} \right] = \frac{3e^{2\pi} - 3 - 2\pi^3}{12}$$

EXAMPLE 5: Find the area of the region bounded by the curves $r = 2\cos \theta$, $r = \cos \theta$, and the rays $\theta = 0$ and $\theta = \pi/4$.

SOLUTION: Here is the graph of this region.

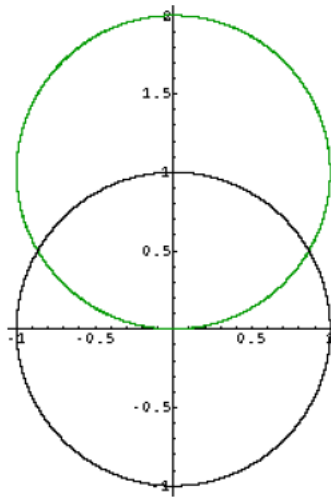


The bounds of integration are $0 \leq \theta \leq \pi/4$ and the outer curve is $r = 2\cos \theta$ and the inner curve is $r = \cos \theta$. Let us find the area.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} (4 \cos^2 \theta - \cos^2 \theta) d\theta \\ &= \int_0^{\pi/4} \frac{3}{2} \cos^2 \theta d\theta = \frac{3}{4} \int_0^{\pi/4} (1 + \cos 2\theta) d\theta \\ &= \frac{3}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/4} = \frac{3\pi + 6}{16} \end{aligned}$$

EXAMPLE 6: Find the area shared by the curves $r = 1$ and $r = 2 \sin \theta$.

SOLUTION: Here is the graph of these two curves.



To find the area of the shared region, I will have to find two separate areas. The first area (A_1) is the area inside the circle $r = 2 \sin \theta$, and the second area (A_2) is the region between the curves $r = 2 \sin \theta$ and $r = 1$. Then I will subtract the second area from the first area to find the area of the shared region. $r = 2 \sin \theta$ is a circle of the form $r = 2a \sin \theta$. Therefore $2a = 2$ or $a = 1$. Remember that a is the radius of the circle. Recall that the formula for finding the area of a circle is πr^2 . Therefore the first area is $\pi(1)^2$ or π . $A_1 = \pi$

To find the area between the curves, I must first find the points of intersection of the two graphs.

$$2 \sin \theta = 1 \rightarrow \sin \theta = 1/2 \rightarrow \theta = \pi/6 \text{ and } \theta = 5\pi/6$$

Since this region is symmetric with respect to the y-axis, therefore θ will start at $\pi/6$ and end at $\pi/2$. I will take 2 times the area. The outer curve is $r = 2 \sin \theta$ and the inner curve is $r = 1$.

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} (4 \sin^2 \theta - 1) d\theta \\ &= \int_{\pi/6}^{\pi/2} \left(4 \left(\frac{1}{2} \right) (1 - \cos 2\theta) - 1 \right) d\theta \\ &= \int_{\pi/6}^{\pi/2} (1 - 2 \cos 2\theta) d\theta = \theta - \sin 2\theta \Big|_{\pi/6}^{\pi/2} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

$$\text{Area of the shared region} = \pi - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

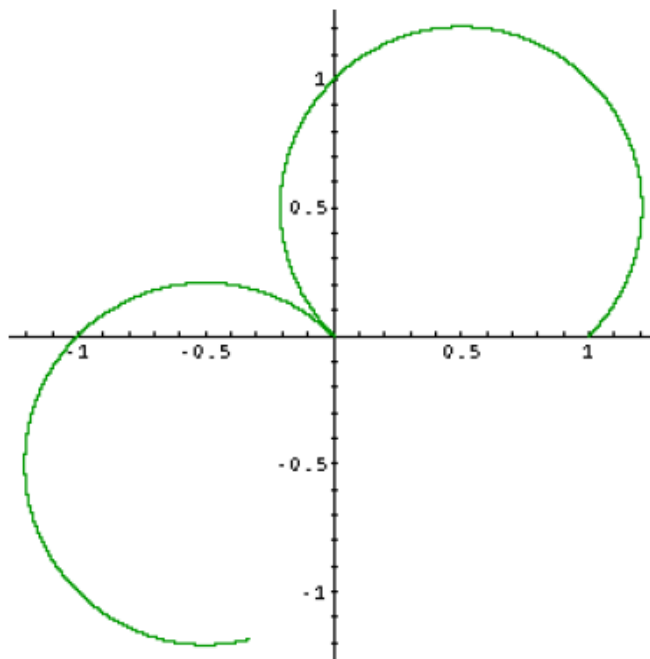
LENGTH OF A CURVE

If $f = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β , then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta.$$

EXAMPLE 7: Find the length of the curve $r = \sqrt{1 + \sin 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$.

SOLUTION: Here is the graph of this curve.



First of all, find $dr/d\theta$.

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{1}{2}(1 + \sin 2\theta)^{-\frac{1}{2}}(\cos 2\theta)(2) \\ &= \frac{\cos 2\theta}{\sqrt{1 + \sin 2\theta}}\end{aligned}$$

Now simplify $r^2 + \left(\frac{dr}{d\theta}\right)^2$.

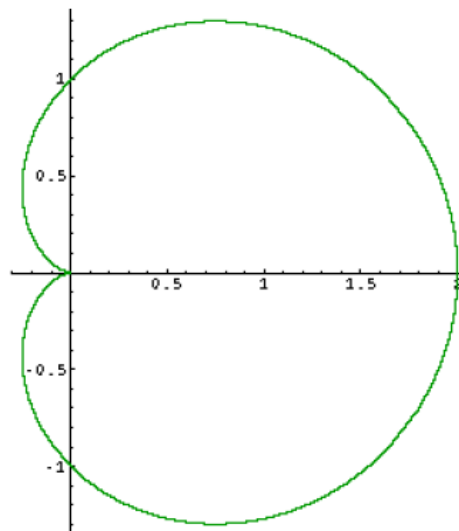
$$\begin{aligned}r^2 + \left(\frac{dr}{d\theta}\right)^2 &= 1 + \sin 2\theta + \frac{\cos^2 2\theta}{1 + \sin 2\theta} \\ &= \frac{(1 + \sin 2\theta)^2 + \cos^2 2\theta}{1 + \sin 2\theta} \\ &= \frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta} \\ &= \frac{2 + 2\sin 2\theta}{1 + \sin 2\theta} = 2\end{aligned}$$

Now find the length.

$$L = \int_0^{\pi\sqrt{2}} \sqrt{2} d\theta = \sqrt{2}\theta \Big|_0^{\pi\sqrt{2}} = 2\pi$$

EXAMPLE 10: Find the length of the curve $r = -1 + \cos \theta$.

SOLUTION: Here is the graph of this curve.



First of all, when tracing out this curve, θ will start at 0 and go to 2π .

Now find $dr/d\theta$.

$$\frac{dr}{d\theta} = -\sin \theta$$

Now simplify $r^2 + \left(\frac{dr}{d\theta}\right)^2$.

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= 1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta \\ &= 2 - 2\cos \theta = 2(1 - \cos \theta) \end{aligned}$$

$$L = \int_0^{2\pi} \sqrt{2(1 - \cos \theta)} d\theta = \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos \theta} d\theta \quad \otimes$$

$$\text{Recall } \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta). \text{ If } \sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta),$$

$$\text{so } 1 - \cos \theta = 2\sin^2 \frac{\theta}{2}.$$

$$\otimes = \int_0^{2\pi} \sqrt{2} \sqrt{2} \sin \frac{\theta}{2} d\theta = 4 \int_0^{\pi} \sin u du = -4 \cos u \Big|_0^{\pi} = 8$$

$$\text{Let } u = \frac{\theta}{2}, du = \frac{d\theta}{2}.$$

AREA OF A SURFACE OF REVOLUTION

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β , then the areas of the surfaces generated by revolving the curve about the x- and y-axis are given by the following formulas:

1. Revolution about the x-axis ($y \geq 0$)

$$S = \int_{\alpha}^{\beta} 2 \pi r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

2. Revolution about the y-axis ($x \geq 0$)

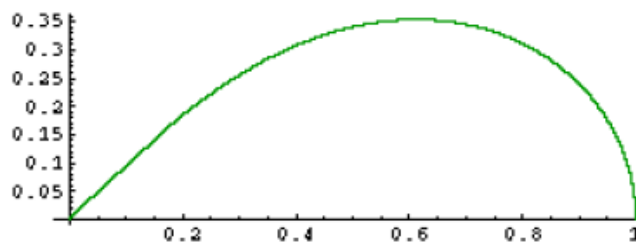
$$S = \int_{\alpha}^{\beta} 2 \pi r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

EXAMPLE 11: Find the area of the surface generated by revolving

$$r = \sqrt{\cos 2\theta} \quad 0 \leq \theta \leq \frac{\pi}{4}$$

about the x-axis.

SOLUTION: Here is the graph of this curve.



$$\frac{dr}{d\theta} = \frac{1}{2} (\cos 2\theta)^{-\frac{1}{2}} (-\sin 2\theta)(2) = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$r^2 + \left(\frac{dr}{d\theta} \right)^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}$$

$$= \frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta} = \frac{1}{\cos 2\theta}$$

$$S = \int_0^{\frac{\pi}{4}} 2\pi \sqrt{\cos 2\theta} \sin \theta \frac{1}{\sqrt{\cos 2\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\pi \sin \theta d\theta = -2\pi \cos \theta \Big|_0^{\frac{\pi}{4}} = 2\pi - \frac{2\pi}{\sqrt{2}}$$

In this set of supplemental notes, I have done several examples of finding the area of a region, area between curves using polar coordinates. I have also done some examples of finding the length of the curve and the surface area of a surface of revolution. The methods are basically the same to what we did in calculus I, but we are now using polar equations to represent the curves.