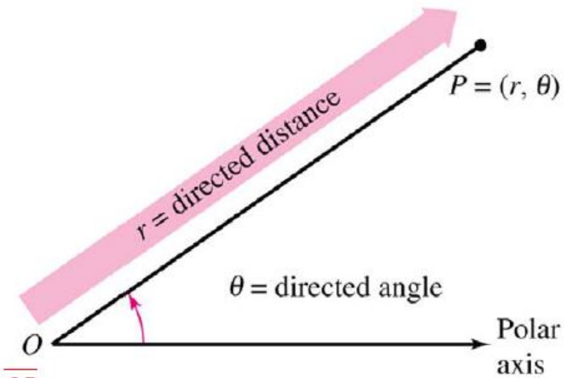
## POLAR COORDINATES

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or *x*-*y*) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system.

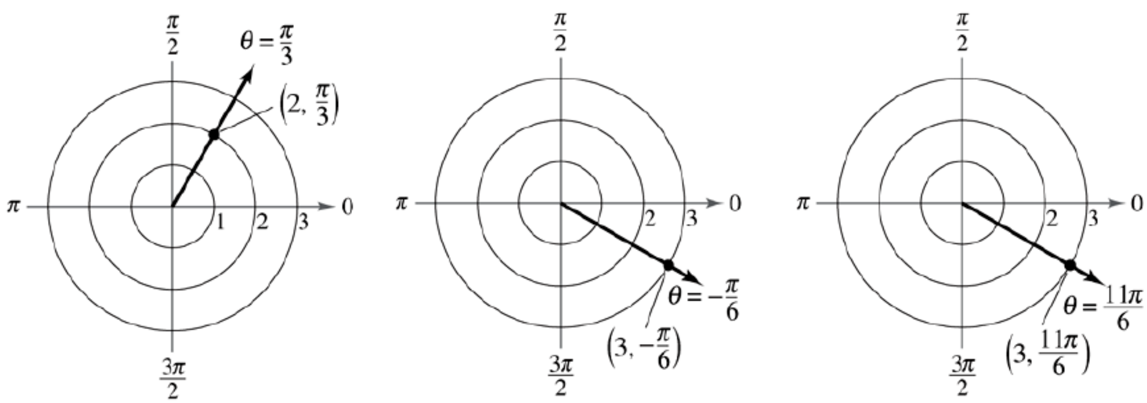
To form the **polar coordinate system** in the plane, fix a point *O*, called the **pole** (or **origin**), and construct from *O* an initial ray called the **polar axis**, as shown in the figure.




Then each point *P* in the plane can be assigned **polar coordinates** $(r,\theta)$, as follows.

$r =$ directed distance from $O$ to $P$

$\theta =$ directed angle, counter clockwise from polar axis to $\overline{OP}$

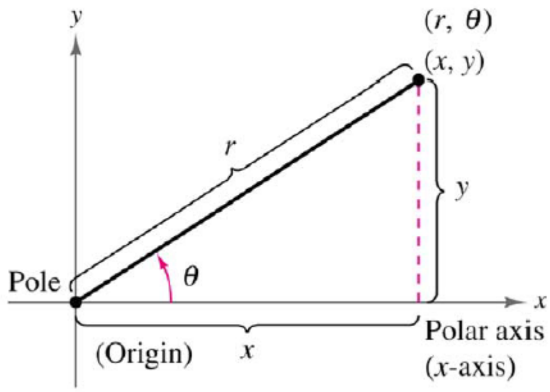



**(a)** **(b)** **(c)**

With rectangular coordinates, each point $(x, y)$ has a unique representation. This is not true with polar coordinates. For instance, the coordinates $(r,\theta)$ and $(r, 2\pi+\theta)$ represent the same point. Also because $r$ is a directed distance, the coordinates $(r,\theta)$ and $(-r, \pi+\theta)$ represent the same point.

## COORDINATES CONVERSION

To establish the relationship between polar and rectangular coordinates , let the polar axis to coincide with the positive *x-axis* and the pole with the origin.




$$\tan(\theta) = \frac{y}{x}$$

$$\cos(\theta) = \frac{x}{r}, \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

### THEOREM 10.10 Coordinate Conversion

The polar coordinates $(r, \theta)$ of a point are related to the rectangular coordinates $(x, y)$ of the point as follows.

**1.** $x = r\cos\theta$ **2.** $\tan\theta = \dfrac{y}{x}$

$y = r\sin\theta$ $r^2 = x^2 + y^2$

**Sample Problem #1:**

Convert each of the following points into the given coordinate system.

a) $\left(-4, \dfrac{2\pi}{3}\right)$ into Cartesian coordinates

b) $(-1,-1)$ into Polar coordinates

**1.** $x = r\cos\theta$

$y = r\sin\theta$

**2.** $\tan\theta = \dfrac{y}{x}$

$r^2 = x^2 + y^2$

Answers

a) $r=-4 \quad \theta = 2\pi/3$

$x = -4\cos(2\pi/3) = 4\cos(\pi/3) = 2$

$y = -4\sin(2\pi/3) = -4\sin(\pi/3) = -2\sqrt{3}$

$\boxed{(2, -2\sqrt{3})}$

b) $(-1,-1)$ is on the III quadrand, and $\tan\theta = 1 \Rightarrow \theta = 5\pi/4$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$\boxed{(\sqrt{2}, 5\pi/4)}$

**Sample Problem #2:**

Convert each of the following into an equation in the given coordinate system.

a) Convert $2x-5x^3=1+xy$ into polar coordinates.

b) Convert $r=-8\cos(\theta)$ into Cartesian coordinates.

Answers

a) $2x - 5x^3 = 1 + xy$

$\boxed{2r\cos\theta - 5r^2\cos^2\theta = 1 + r^2\cos\theta\sin\theta}$

b) $r = -8\cos(\theta)$ (multiply by r)

$r^2 = -8r\cos(\theta)$

$x^2 + y^2 = -8x$

$x^2 + 8x + y^2 = 0$

$x^2 + 8x + 16 + y^2 = 16$

$\boxed{(x+4)^2 + y^2 = 16}$

## COMMON POLAR GRAPHS

**LINES:** Some lines have fairly simple equations in polar coordinates.

1. $\theta = \beta$

We can see that this is a line by converting to Cartesian coordinates as follows

$$\theta = \beta$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \beta$$

$$\frac{y}{x} = \tan\beta$$

$$y = (\tan\beta)\,x$$

This is a line that goes through the origin and makes an angle of $\beta$ with the positive $x$-axis. Or, in other words it is a line through the origin with slope of $\tan\beta$.

2. $r\cos(\theta) = a$

This is easy enough to convert to Cartesian coordinates to $x = a$. So, this is a vertical line.

3. $r\sin(\theta) = b$

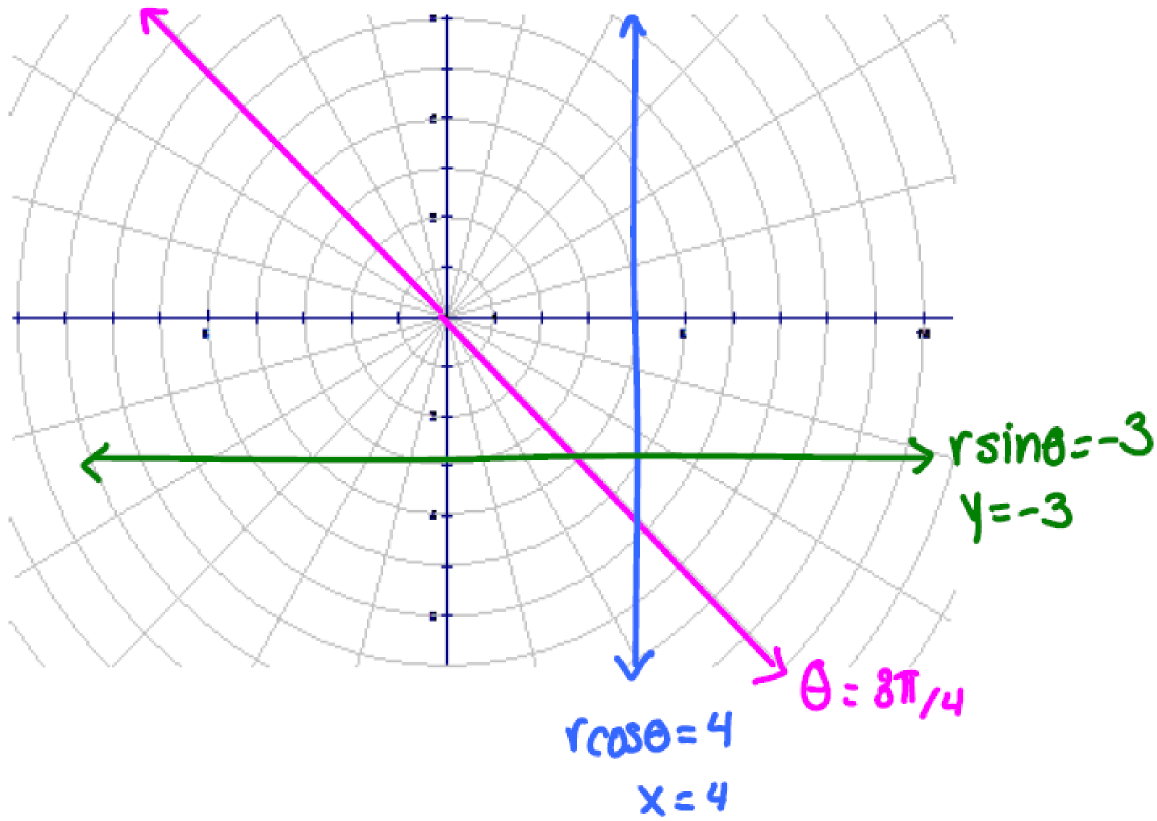
Likewise, this converts to $y = b$ and so is a horizontal line.

**Sample Problem #3:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $\theta = \frac{3\pi}{4}$, $r\cos\theta = 4$ and $r\sin\theta = -3$ on the same axis system.

# Answers

Graph $\theta = \dfrac{3\pi}{4}$, $r\cos\theta = 4$ and $r\sin\theta = -3$ on the same axis system.



**CIRCLES:**

Let's take a look at the equations of circles in polar coordinates.

1. $r = a$.
   This equation is saying that no matter what angle we've got the distance from the origin must be $a$. If you think about it that is exactly the definition of a circle of radius $a$ centered at the origin.

   So, this is a circle of radius $a$ centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the corresponding equation in Cartesian coordinates.

2. $r = 2a\cos\theta$.
   We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

   This is a circle of radius $|a|$ and center $(a,0)$. Note that $a$ might be negative (as it was in our example above) and so the absolute value bars are required on the radius. They should not be used however on the center.

3. $r = 2b\sin\theta$.
   This is similar to the previous one. It is a circle of radius $|b|$ and center $(0,b)$.

4. $r = 2a\cos\theta + 2b\sin\theta$.
   This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2+b^2}$ and center $(a,b)$. In other words, this is the general equation of a circle that isn't centered at the origin.

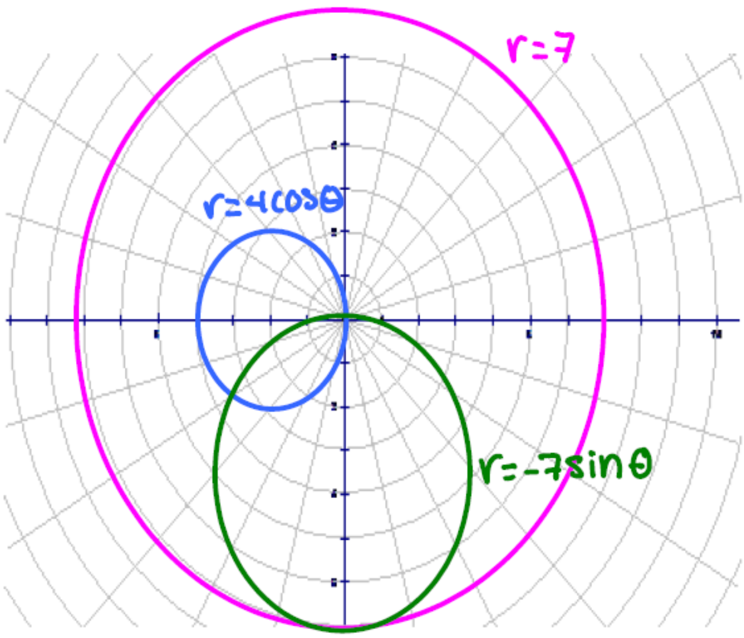
**Sample Problem #4:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $r = 7$, $r = 4\cos\theta$, and $r = -7\sin\theta$ on the same axis system.

## Answers

**Sample Problem #4:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $r = 7$, $r = 4\cos\theta$, and $r = -7\sin\theta$ on the same axis system.




$r = 4\cos\theta$

$r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 4$

$r = -7\sin\theta$

$r^2 = -7r\sin\theta$

$x^2 + y^2 = -7y$

$x^2 + y^2 + 7y + \frac{49}{4} = \frac{49}{4}$

$x^2 + (y + 7/2)^2 = \frac{49}{4}$

**CARDIOIDS and LIMACONS:**

These can be broken up into the following three cases.

1. Cardioids : $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$.
   These have a graph that is vaguely heart shaped and always contain the origin.

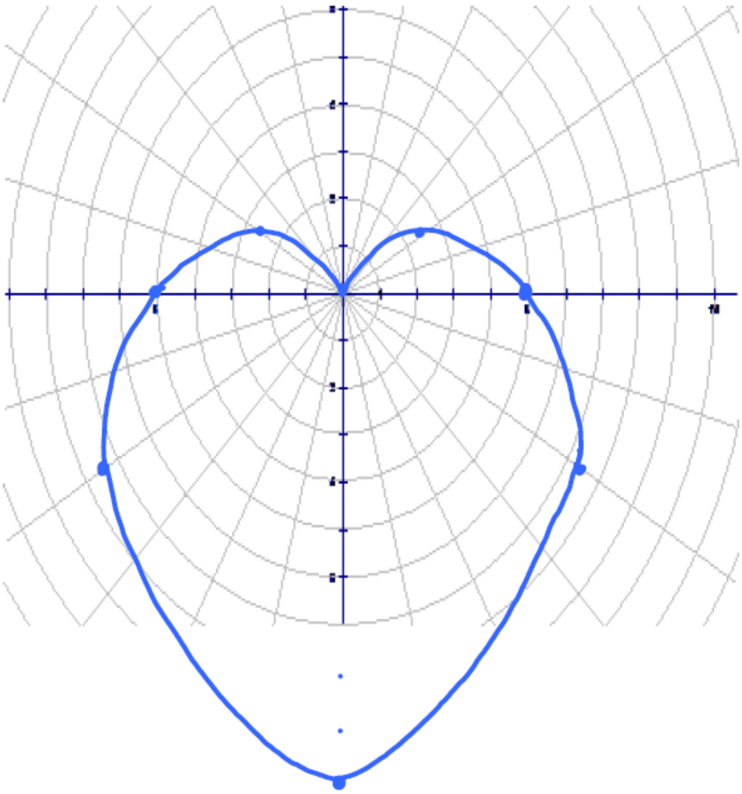
2. Limacons with an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a < b$.
   These will have an inner loop and will always contain the origin.

3. Limacons without an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a > b$.
   These do not have an inner loop and do not contain the origin.

Answers

**Sample Problem #5:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $r = 5 - 5\sin\theta$ , $r = 7 - 6\cos\theta$ , and $r = 2 + 4\cos\theta$



$r = 5 - 5\sin\theta$

| $\theta$ | $r$ |
|---|---|
| $0$ | $5$ |
| $\pi/6$ | $2.5$ |
| $\pi/2$ | $0$ |
| $7\pi/6$ | $15/2$ |
| $3\pi/2$ | $10$ |

$\pi$
$5\pi/6$ } Symmetry

$11\pi/6$ Symmetry

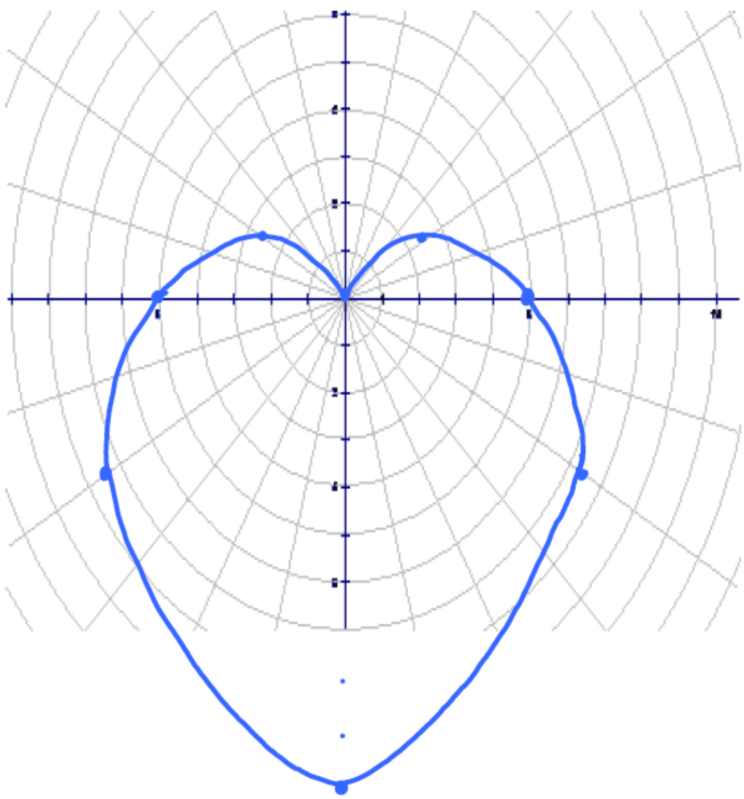
**Sample Problem #5:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $r = 5 - 5\sin\theta$ , $r = 7 - 6\cos\theta$ , and $r = 2 + 4\cos\theta$

Answers

**Sample Problem #5:** **USE YOUR POLAR COORDINATE GRAPH PAPER**

Graph $r = 5 - 5\sin\theta$, $r = 7 - 6\cos\theta$, and $r = 2 + 4\cos\theta$



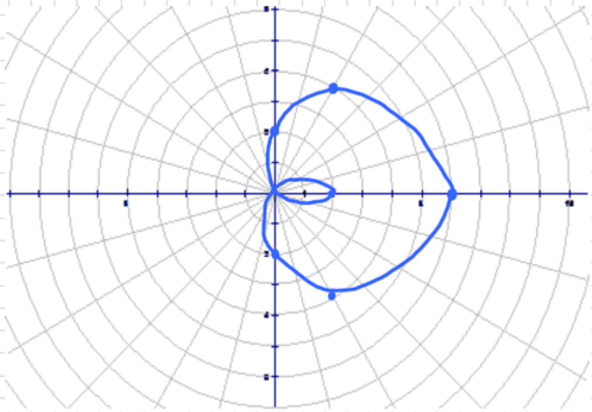
$r = 5 - 5\sin\theta$

| $\theta$ | $r$ | |
|---|---|---|
| $0$ | $5$ | $\pi$ |
| $\pi/6$ | $2.5$ | $5\pi/6$ |
| $\pi/2$ | $0$ | |
| $7\pi/6$ | $15/2$ | $11\pi/6$ |
| $3\pi/2$ | $10$ | |

Symmetry

Symmetry

$r = 2 + 4\cos\theta$



| $\theta$ | $r$ |
|---|---|
| $0$ | $6$ |
| $\pi/3$ | $4$ |
| $\pi/2$ | $2$ |
| $2\pi/3$ | $0$ |
| $\pi$ | $-2$ |

III and IV use symmetry of graph.

## TANGENTS WITH POLAR COORDINATES

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with $\theta$ as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta\end{aligned} \qquad\qquad \begin{aligned}\frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \\ &= \frac{dr}{d\theta}\sin\theta + r\cos\theta\end{aligned}$$

The derivative $\frac{dy}{dx}$ is then,

**Derivative with Polar Coordinates**

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Note that rather than trying to remember this formula it would probably be easier to remember how we derived it and just remember the formula for parametric equations.

**Sample Problem #5:**

Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

## Answers

**Sample Problem #5:**

Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

$$\begin{cases} y = r\sin\theta \\ x = r\cos\theta \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = 8\cos\theta$$

$$\left.\frac{dr}{d\theta}\right|_{\pi/6} = 8\sqrt{3}/2 = 4\sqrt{3}$$

$$r(\pi/6) = 7$$

$$\frac{dy}{dx} = \frac{4\sqrt{3}\cdot\frac{1}{2} + 7\cdot\frac{\sqrt{3}}{2}}{4\sqrt{3}\cdot\frac{\sqrt{3}}{2} - 7\cdot\frac{1}{2}} = \frac{\frac{11\sqrt{3}}{\cancel{2}}}{\frac{5}{\cancel{2}}}$$

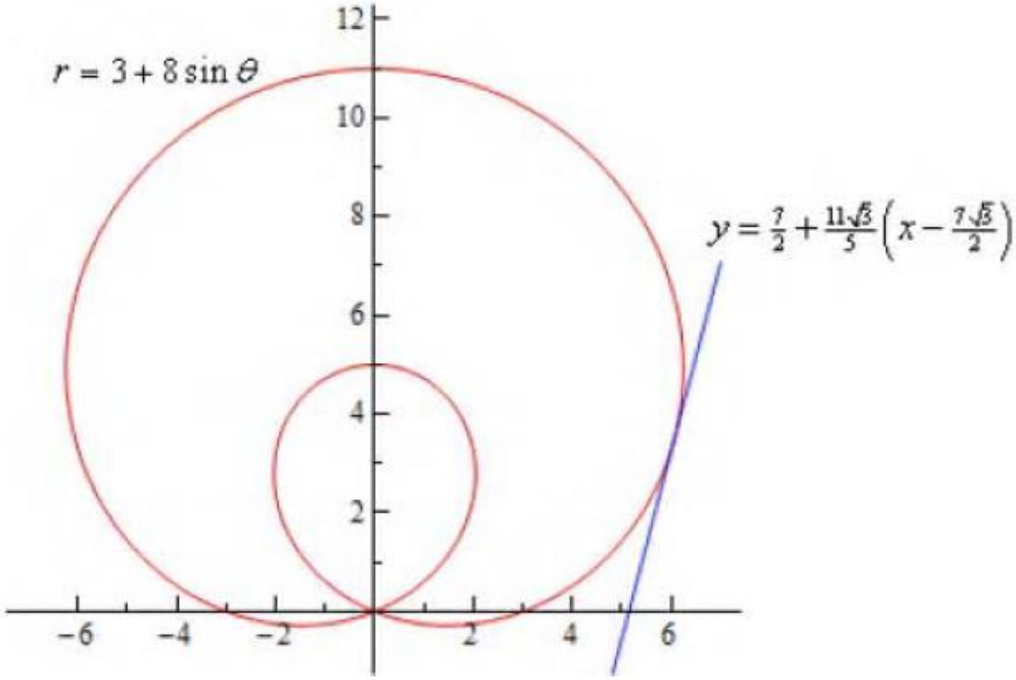
$$\frac{dy}{dx} = 11\sqrt{3}/5$$

$$\left.\begin{aligned} m &= \frac{11\sqrt{3}}{5} \\ x &= 7\sqrt{3}/2 \\ y &= 7/2 \end{aligned}\right\}$$

**Tangent**

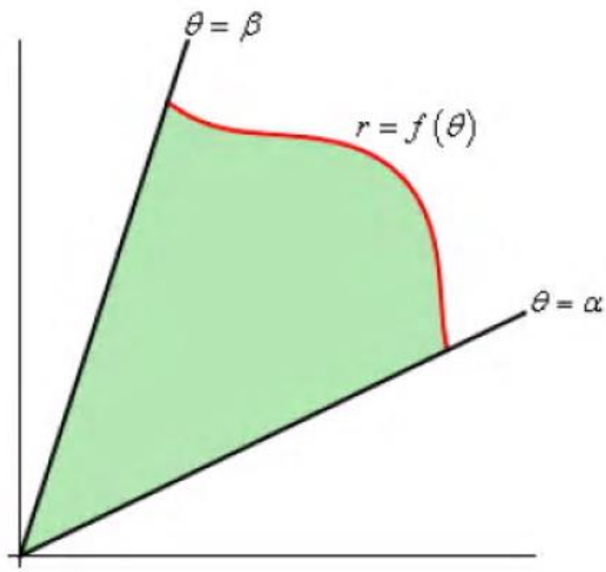
$$\boxed{y - 7/2 = \frac{11\sqrt{3}}{5}\left(x - 7\sqrt{3}/2\right)}$$

For the sake of completeness here is a graph of the curve and the tangent line.



## AREA WITH POLAR COORDINATES

In this section we are going to look at areas enclosed by polar curves. Note as well that we said "enclosed by" instead of "under" as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like.



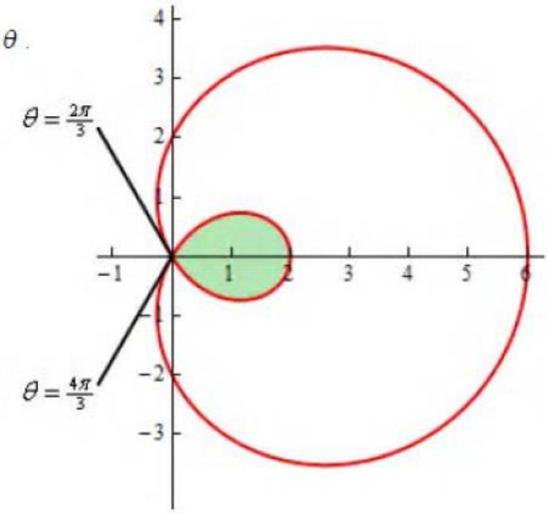

We'll be looking for the shaded area in the sketch above. The formula for finding this area is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Notice that we use $r$ in the integral instead of $f(\theta)$ so make sure and substitute accordingly when doing the integral.

**Sample Problem #6:**

Determine the area of the inner loop of $r = 2 + 4\cos\theta$.




## Answers

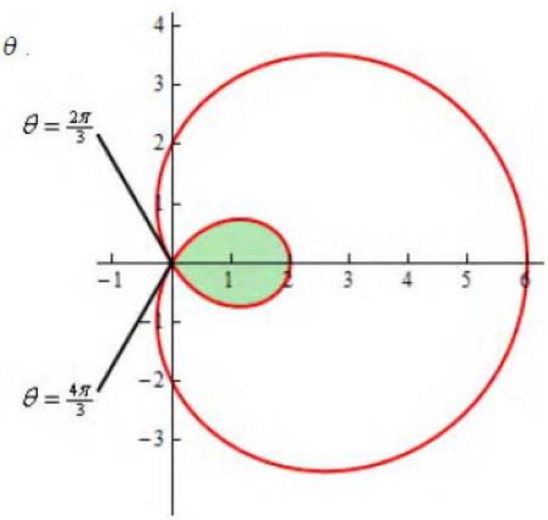
**Sample Problem #6:**

Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

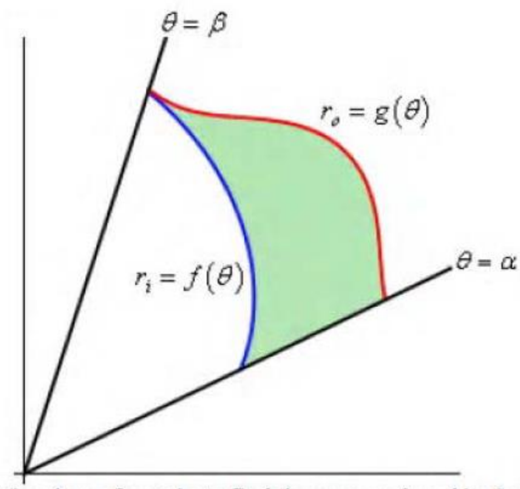
$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (2+4\cos\theta)^2 \, d\theta$$

$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} 4 + 16\cos\theta + \cos^2\theta \, d\theta$$

$$\boxed{A \approx 2.174}$$



So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch were we want to find the area between two curves.



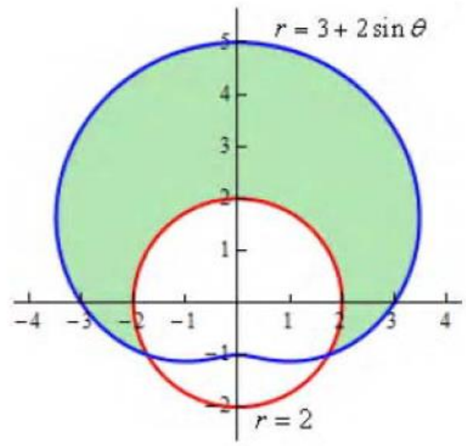

In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta$$

## Answers

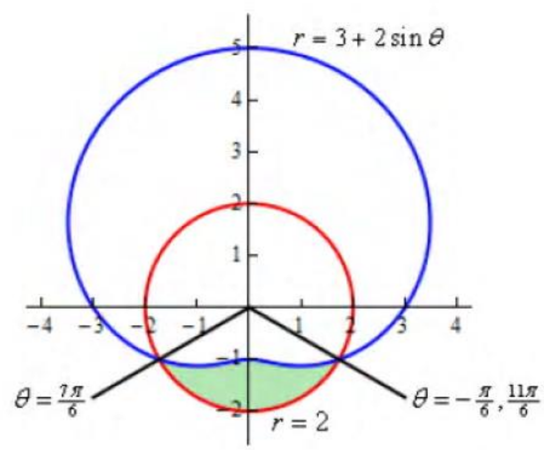
**Sample Problem #7:**

Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.




**Sample Problem #8:**

Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.



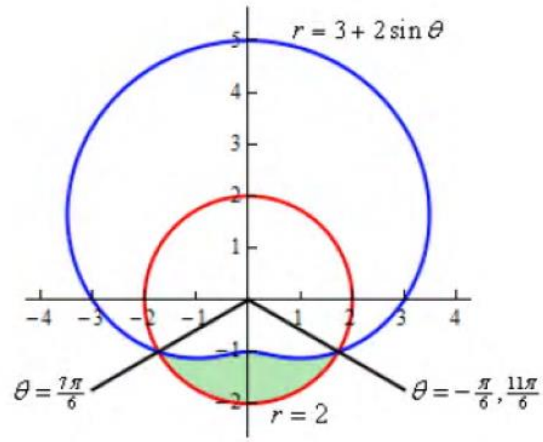
## Answers

**Sample Problem #8:**

Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

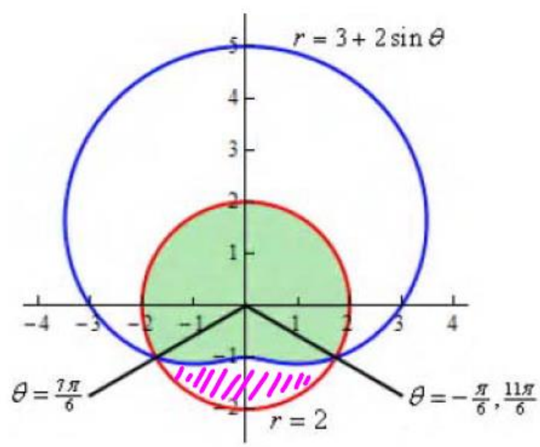
$$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} 4 - (3+2\sin\theta)^2 \, d\theta$$

$$\boxed{A \approx 2.196}$$




**Sample Problem #9:**

Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.



## Answers

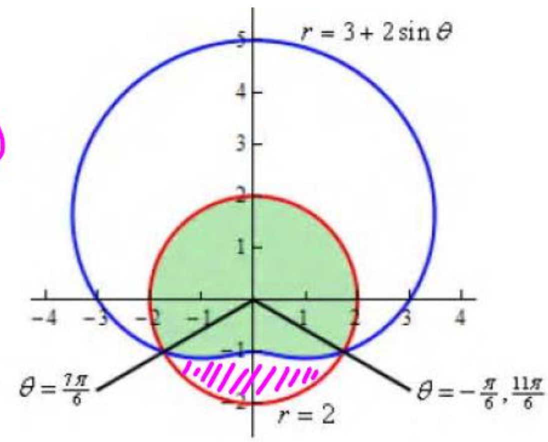
**Sample Problem #9:**

Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

$A_{\odot} = 4\pi$

$A_{pink} \approx 2.196$ (from above question)

$\boxed{A \approx 10.370}$




## ARC LENGTH WITH POLAR COORDINATES

The arc length formula for polar coordinates is:

$$L = \int ds$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

**Sample Problem #10:**

Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

Answers

**Sample Problem #10:**

Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

$\frac{dr}{d\theta} = 1 \quad ds = \sqrt{\theta^2 + 1} \qquad L = \int_0^1 \sqrt{\theta^2+1}\, d\theta \approx \boxed{1.148}$

## SURFACE AREA WITH POLAR COORDINATES

As we did in the tangent and arc length sections we'll write the curve in terms of a set of parametric equations.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
$$= f(\theta)\cos\theta \qquad\qquad = f(\theta)\sin\theta$$

If we now use the parametric formula for finding the surface area we'll get,

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$
$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad r = f(\theta),\quad \alpha \le \theta \le \beta$$

## HOMEWORK:

**SECTION 1:**

For each of the following polar coordinates, find the corresponding rectangular coordinates. $(-2, 2\sqrt{3})$

1. $\left(6, \frac{\pi}{2}\right)$ $(0, 6)$
2. $\left(-1, \frac{7\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. $\left(-4, \frac{-\pi}{3}\right)$

For each of the following rectangular coordinates, find two corresponding polar coordinates.

4. (-3, 3) $(3, 3\pi/4)$ $(-3, 7\pi/4)$
5. $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$ $(1, -\pi/3)$ $(-1, 2\pi/3)$
6. (0, -4) $(4, 3\pi/2)$ $(-4, \pi/2)$

For each of the following rectangular equations, change it to polar form and confirm on your calculator.

7. $5x - y = 7$

$\boxed{5r\cos\theta - r\sin\theta = 7}$

8. $xy = 12$

$\boxed{r^2\sin\theta\cos\theta = 12}$

9. $(x-1)^2 + y^2 = 1$

$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$r^2 - 2r\cos\theta + 1 = 1$

$r^2 = 2r\cos\theta$ $\boxed{r = 2\cos\theta}$

10. $x^2 + y^2 + 4x = 0$

$1 + 4r\cos\theta = 0$

$\boxed{r\cos\theta = -1/4}$

For each of the following polar equations, change it to rectangular form and confirm on your calculator.

11. $r = 4$

$\boxed{x^2 + y^2 = 16}$

12. $\tan^2\theta = 9$

$\frac{\sin^2\theta}{\cos^2\theta} = 9$ $\frac{y^2}{x^2} = 9$ $y^2 = 9x^2$

$\boxed{y = \pm 3x}$

13. $r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y + 4 = 4$

$\boxed{x^2 + (y-2)^2 = 4}$

14. $r = \frac{1}{1 - \cos\theta}$

$r - r\cos\theta = 1$

$r = r\cos\theta$

$r^2 = r^2\cos^2\theta$

$x^2 + y^2 = x^2$ $y^2 = 0$ $\boxed{y = 0}$

# HOMEWORK:

**SECTION 1:**

Answers

For each of the following polar coordinates, find the corresponding rectangular coordinates.

1. $\left(6, \frac{\pi}{2}\right)$ $(0, 6)$
2. $\left(-1, \frac{7\pi}{4}\right)$ $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. $\left(-4, \frac{-\pi}{3}\right)$ $(-2, 2\sqrt{3})$

For each of the following rectangular coordinates, find two corresponding polar coordinates.

4. (-3, 3) $(3, 3\pi/4)$ $(-3, 7\pi/4)$
5. $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$ $(1, -\pi/3)$ $(-1, 2\pi/3)$
6. (0, -4) $(4, 3\pi/2)$ $(-4, \pi/2)$

For each of the following rectangular equations, change it to polar form and confirm on your calculator.

7. $5x - y = 7$

$\boxed{5r\cos\theta - r\sin\theta = 7}$

8. $xy = 12$

$\boxed{r^2\sin\theta\cos\theta = 12}$

9. $(x-1)^2 + y^2 = 1$

$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$r^2 - 2r\cos\theta + 1 = 1$

$r^2 = 2r\cos\theta$ $\boxed{r = 2\cos\theta}$

10. $x^2 + y^2 + 4x = 0$

$1 + 4r\cos\theta = 0$

$\boxed{r\cos\theta = -1/4}$

For each of the following polar equations, change it to rectangular form and confirm on your calculator.

11. $r = 4$

$\boxed{x^2 + y^2 = 16}$

12. $\tan^2\theta = 9$

$\frac{\sin^2\theta}{\cos^2\theta} = 9$ $\frac{y^2}{x^2} = 9$ $y^2 = 9x^2$

$\boxed{y = \pm 3x}$

13. $r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y + 4 = 4$

$\boxed{x^2 + (y-2)^2 = 4}$

14. $r = \frac{1}{1 - \cos\theta}$

$r - r\cos\theta = 1$

$r = r\cos\theta$

$r^2 = r^2\cos^2\theta$

$x^2 + y^2 = x^2$ $y^2 = 0$ $\boxed{y = 0}$

For $r = 2 + 3\sin\theta$, find $\frac{dy}{dx}$ and the slope of the tangent line at the following polar points.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$y = r\sin\theta$

$x = r\cos\theta$

$\frac{dr}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 2\cos\theta + 3\cos\theta\sin\theta}{3\cos^2\theta - 2\sin\theta - 3\sin^2\theta} = \frac{6\cos\theta\sin\theta}{3\cos^2\theta - 2\sin\theta - 3\sin^2\theta}$$

15. $\left(5, \frac{5\pi}{2}\right)$

$\frac{dy}{dx} = 0$

16. $\left(-1, \frac{3\pi}{2}\right)$

$\frac{dy}{dx} = 0$

17. $(2, \pi)$

$\frac{dy}{dx} = 0$

For each of the following, find the points of horizontal and vertical tangency (if any)

18. $r = 3 + \sin\theta$

at end, need space!

19. $r = \sin\theta\cos^2\theta, \qquad 0 \le \theta < \pi$

at end, need space!

Find the lines tangents at the pole to

20. $r = 4(1 - \cos\theta)$

21. $r = 3\sin 2\theta$

at the end, need space

Answers

For $r = 2 + 3\sin\theta$, find $\frac{dy}{dx}$ and the slope of the tangent line at the following polar points.

$y = r\sin\theta$

$x = r\cos\theta$

$\frac{dr}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + 2\cos\theta + 3\cos\theta\sin\theta}{3\cos^2\theta - 2\sin\theta - 3\sin^2\theta} = \frac{6\cos\theta\sin\theta}{3\cos^2\theta - 2\sin\theta - 3\sin^2\theta}$$

15. $\left(5, \frac{5\pi}{2}\right)$ $\frac{dy}{dx} = 0$

16. $\left(-1, \frac{3\pi}{2}\right)$ $\frac{dy}{dx} = 0$

17. $(2, \pi)$ $\frac{dy}{dx} = 0$

For each of the following, find the points of horizontal and vertical tangency (if any)

18. $r = 3 + \sin\theta$

at end, need space!

19. $r = \sin\theta\cos^2\theta, \quad 0 \le \theta < \pi$

at end, need space!

Find the lines tangents at the pole to

20. $r = 4(1 - \cos\theta)$

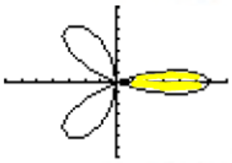
21. $r = 3\sin 2\theta$

at the end, need space

## SECTION 2:

In the following, graph on your calculator and find the area of the region. Set up and use the calculator.

1. One petal of $r = 6\cos 3\theta$



| $\theta$ | $r$ |
|---|---|
| 0 | 6 |
| $\pi/6$ | 0 |

$$A = 2\int_0^{\pi/6} \tfrac{1}{2}(6\cos^2 3\theta)\,d\theta$$

$$A = 36\int_0^{\pi/6} \cos^2(3\theta)\,d\theta \approx \boxed{9.425}$$

2. One petal of $r = 2\sin 2\theta$

At end, need space!

3. One petal of $r = \sin 5\theta$

4. Interior of $r = 2 - \sin\theta$ (above polar axis)

at end, need space!

5. Inner loop of $r = 1 - 2\cos\theta$

6. Between the loops of $r = 1 + 2\cos\theta$

at end, need space!

Find the points of intersection of the graphs of

7. $r = 2 + \cos\theta$ and $r = 2 - \cos\theta$

$2 + \cos\theta = 2 - \cos\theta$

$\cos\theta = -\cos\theta$

$\cos\theta = 0$

$\boxed{\theta = \pi/2,\ 3\pi/2}$

8. $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$

$1 + \cos\theta = 1 - \sin\theta$

$\cos\theta = -\sin\theta$

$\tan\theta = -1$

$\boxed{\theta = 3\pi/4,\ 7\pi/4}$

9. $r = 1 + \cos\theta$ and $r = 3\cos\theta$

$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = 1/2$

$\boxed{\theta = \pi/3,\ 5\pi/3}$

10. $r = 4\sin 2\theta$ and $r = 2$

$4\sin(2\theta) = 2$

$\sin(2\theta) = 1/2$

$2\theta = \pi/6 + 2n\pi = \dfrac{\pi + 12n\pi}{6}$

$2\theta = 5\pi/6 + 2n\pi = \dfrac{5\pi + 12n\pi}{6}$

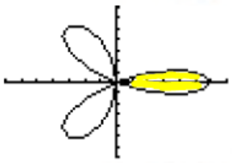
$\theta = \dfrac{\pi + 12n\pi}{12}$ $\quad \theta = \dfrac{5\pi + 12n\pi}{12}$

$\boxed{\begin{array}{l}\theta = \pi/12 \\ \theta = 5\pi/12 \\ \theta = 13\pi/12 \\ \theta = 17\pi/12\end{array}}$

**SECTION 2:** Answers

In the following, graph on your calculator and find the area of the region. Set up and use the calculator.

1. One petal of $r = 6\cos 3\theta$



| $\theta$ | $r$ |
|---|---|
| 0 | 6 |
| $\pi/6$ | 0 |

$A = 2\int_0^{\pi/6} \frac{1}{2}(6\cos^2 3\theta)\,d\theta$

$A = 36\int_0^{\pi/6} \cos^2(3\theta)\,d\theta \approx \boxed{9.425}$

2. One petal of $r = 2\sin 2\theta$

At end, need space!

3. One petal of $r = \sin 5\theta$

4. Interior of $r = 2 - \sin\theta$ (above polar axis)

at end, need space!

5. Inner loop of $r = 1 - 2\cos\theta$

6. Between the loops of $r = 1 + 2\cos\theta$

at end, need space!

Find the points of intersection of the graphs of

7. $r = 2 + \cos\theta$ and $r = 2 - \cos\theta$

$2 + \cos\theta = 2 - \cos\theta$

$\cos\theta = -\cos\theta$

$\cos\theta = 0$

$\boxed{\theta = \pi/2,\ 3\pi/2}$

8. $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$

$1 + \cos\theta = 1 - \sin\theta$

$\cos\theta = -\sin\theta$

$\tan\theta = -1$

$\boxed{\theta = 3\pi/4,\ 7\pi/4}$

9. $r = 1 + \cos\theta$ and $r = 3\cos\theta$

$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = 1/2$

$\boxed{\theta = \pi/3,\ 5\pi/3}$

10. $r = 4\sin 2\theta$ and $r = 2$

$4\sin(2\theta) = 2$

$\sin(2\theta) = 1/2$

$2\theta = \pi/6 + 2n\pi = \dfrac{\pi + 12n\pi}{6}$

$2\theta = 5\pi/6 + 2n\pi = \dfrac{5\pi + 12n\pi}{6}$

$\theta = \dfrac{\pi + 12n\pi}{12}$ $\quad\theta = \dfrac{5\pi + 12n\pi}{12}$

$\boxed{\begin{array}{l}\theta = \pi/12\\ \theta = 5\pi/12\\ \theta = 13\pi/12\\ \theta = 17\pi/12\end{array}}$

Use your calculators to graph the polar equations and find the area of the indicated region.

11. Common interior of $r = 4(1+\sin\theta)$ and $r = 4(1-\sin\theta)$

12. Outside $r = 2\sin\theta$ and inside $r = 2$

13. Inside the lemniscate of $r = \sqrt{25\cos 2\theta}$

14. Radiation from an antenna is modeled by $r = a\cos^2\theta$. Find the area of the geographical region between the two curves for $a = 4$ and $a = 6$

Use your calculators to graph the polar equations and find the length of the curve accurate to 3 decimal places.

15. $r = 2 + \sin\theta \qquad 0 \le \theta \le 2\pi$

16. $r = 2\theta \qquad 0 \le \theta \le \frac{\pi}{2}$

17. $r = \sec\theta \qquad 0 \le \theta \le \frac{\pi}{4}$ (No calculator)

18. $r = e^{\theta} \qquad 0 \le \theta \le \pi$ (No calculator)

Use your calculators to graph the polar equations and find the surface area when rotated about the given line.

19. $r = 2\sin\theta \quad 0 \le \theta \le \frac{\pi}{2} \quad \theta = \frac{\pi}{2}$ (No calculator)

20. $r = e^{-\theta} \qquad 0 \le \theta \le \frac{\pi}{2} \qquad \theta = 0$

18. $r = 3 + \sin\theta$

**Horizontal Tangent**

$\frac{dy}{d\theta} = 0 \qquad y = r\sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$

$\cos\theta\sin\theta + (3+\sin\theta)\cos\theta = 0$

$\cos\theta(\sin\theta + 3 + \sin\theta) = 0$

$\cos\theta(3 + 2\sin\theta) = 0$

$\cos\theta = 0$

$\Rightarrow$ Horizontal Tangent Line at $\theta = (2n+1)\frac{\pi}{2}$ (n-intg.)

$$\boxed{\left(3+(-1)^n, (2n+1)\pi/2\right)}$$

**Vertical Tangent**

$\frac{dx}{d\theta} = 0 \qquad x = r\cos\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$

$\cos\theta\cos\theta - (3+\sin\theta)\sin\theta = 0$

$\cos^2\theta - 3\sin\theta - \sin^2\theta = 0$

$1 - \sin^2\theta - 3\sin\theta - \sin^2\theta = 0$

$-2\sin^2\theta - 3\sin\theta + 1 = 0$

$2\sin^2\theta + 3\sin\theta - 1 = 0$

$\sin\theta = \frac{-3 \pm \sqrt{9-4(2)(-1)}}{2(2)}$

$\sin\theta = \frac{-3 \pm \sqrt{17}}{4}$

$\theta = \arcsin\left(\frac{-3 \pm \sqrt{17}}{4}\right)$

$r = 3 + \frac{-3 \pm \sqrt{17}}{4} = \frac{9 \pm \sqrt{17}}{4}$

$$\boxed{\left(\frac{9 \pm \sqrt{17}}{4}, \arcsin\left(\frac{-3 \pm \sqrt{17}}{4}\right)\right)}$$

19. $r = \sin\theta\cos^2\theta, \qquad 0 \le \theta < \pi$

## Horizontal Tangent

$\frac{dy}{d\theta} = 0 \qquad y = r\sin\theta$

$\frac{dr}{d\theta} = \cos\theta\cos^2\theta - \sin\theta \cdot 2\cos\theta\sin\theta$

$\frac{dr}{d\theta} = \cos^3\theta - 2\sin^2\theta\cos\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$

$\frac{dy}{d\theta} = \cos^3\theta - 2\sin^3\theta\cos\theta + \sin\theta\cos^3\theta = 0$

$\cos\theta\left(\cos^2\theta - 2\sin^3\theta + \sin\theta\cos^2\theta\right) = 0$

$\cos\theta\left(1 - \sin^2\theta - 2\sin^3\theta + \sin\theta(1 - \sin^2\theta)\right) = 0$

$\cos\theta\left(1 - \sin^2\theta - 2\sin^3\theta + \sin\theta - \sin^3\theta\right) = 0$

$\cos\theta\underbrace{\left(-3\sin^3\theta - \sin^2\theta + \sin\theta + 1\right)}_{\theta = 0.827,\ \theta = 2.315} = 0$

$\downarrow$

$\theta = \pi/2$

## Vertical Tangent

$\frac{dx}{d\theta} = 0 \qquad x = r\cos\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$

$\cos^4\theta - 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta = 0$

$\cos^4\theta - 3\sin^2\theta\cos^2\theta = 0$

$\cos^2\theta\left[\cos^2\theta - 3\sin^2\theta\right] = 0$

$\cos^2\theta\underbrace{\left[1 - 4\sin^2\theta\right]}_{\sin^2\theta = 1/4} = 0$

$\downarrow$

$\theta = \pi/2$

$\sin\theta = \pm 1/2$

$\sin\theta = 1/2$

$\theta = \pi/6 \qquad \theta = 5\pi/6$

$$\boxed{\begin{array}{l}(0, \pi/2)\\ (3/4, \pi/6)\\ (3/4, 5\pi/6)\end{array}}$$

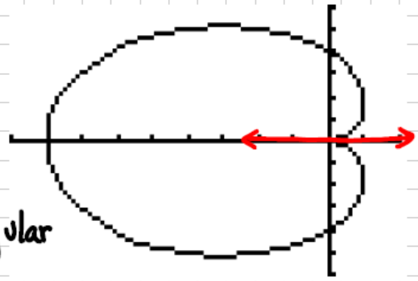
$(0, \pi/2)$

$(0.337, 0.827)$

$(0.337, 2.315)$

Find the lines tangents at the pole to

20. $r = 4(1-\cos\theta)$



$\theta = 0 \Rightarrow r = 0 \Rightarrow (0,0) \leftarrow$ polar

$(0,0) \leftarrow$ rectangular

$y = r\sin\theta$

$y = 4(1-\cos\theta)\sin\theta = 4\sin\theta - 4\sin\theta\cos\theta$

$y = 4\sin\theta - 2\sin(2\theta)$

$\frac{dy}{d\theta} = 4\cos\theta - 4\cos(2\theta)$

$\left.\frac{dy}{d\theta}\right|_{\theta=0} = 0$

$x = r\cos\theta$

$x = 4(1-\cos\theta)\cos\theta$

$x = 4\cos\theta - 4\cos^2\theta$

$\frac{dx}{d\theta} = -4\sin\theta + 8\cos\theta\sin\theta$

$\left.\frac{dx}{d\theta}\right|_{\theta=0} = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4(\cos\theta - \cos(2\theta))}{4(2\sin(2\theta) - \sin\theta)} = \frac{\cos\theta - (2\cos^2\theta - 1)}{4\sin\theta\cos\theta - \sin\theta}$$

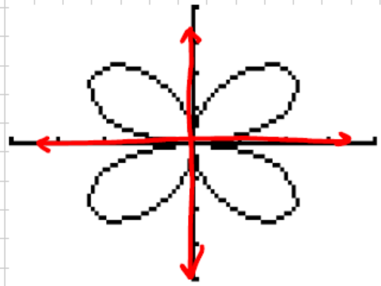
$$\frac{dy}{dx} = -\frac{2\cos^2\theta - \cos\theta - 1}{\sin\theta(4\cos\theta - 1)} = -\frac{(\cos\theta - 1)(2\cos\theta + 1)}{\sin\theta(4\cos\theta - 1)}$$

$$\frac{dy}{dx} = \frac{(1-\cos\theta)(2\cos\theta+1)}{\sin\theta(4\cos\theta-1)} \cdot \frac{(1+\cos\theta)}{(1+\cos\theta)} = \frac{(1-\cos^2\theta)(2\cos\theta+1)}{\sin\theta(4\cos\theta-1)(1+\cos\theta)}$$

$$\frac{dy}{dx} = \frac{\sin^{\cancel{2}}\theta(2\cos\theta+1)}{\cancel{\sin\theta}(4\cos\theta-1)(1+\cos\theta)} = \frac{\sin\theta(2\cos\theta+1)}{(4\cos\theta-1)(1+\cos\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = 0 \Rightarrow \boxed{\begin{array}{c}\underline{\text{Horizontal}}\ \text{Tangent Line} \\ y=0\end{array}}$$

21. $r = 3\sin 2\theta$



$r=0 \Rightarrow \sin(2\theta)=0$

$2\theta = 0 \longrightarrow \theta = 0$

$2\theta = \pi \longrightarrow \theta = \pi/2$

$2\theta = 2\pi \longrightarrow \theta = \pi$

$2\theta = 3\pi \longrightarrow \theta = 3\pi/2$

$y = r\sin\theta = 3\sin(2\theta)\sin\theta = 6\sin^2\theta\cos\theta$

$\frac{dy}{d\theta} = 12\sin\theta\cos^2\theta - 6\sin^3\theta$

$x = r\cos\theta = 3\sin(2\theta)\cos\theta = 6\sin\theta\cos^2\theta$

$\frac{dx}{d\theta} = 6\cos^3\theta - 6\sin\theta^2 \cdot 2\cos\theta$

$\frac{dx}{d\theta} = 6\cos^3\theta - 12\sin^2\theta\cos\theta$

$\frac{dy}{dx} = \frac{\cancel{6}\sin\theta(2\cos^2\theta - \sin^2\theta)}{\cancel{6}\cos\theta(\cos^2\theta - 2\sin^2\theta)}$

$\frac{dy}{dx} = \frac{\sin\theta(2\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - 2\sin^2\theta)}$

$\theta = 0$

$\theta = \pi/2$

$\theta = \pi$

$\theta = 3\pi/2$

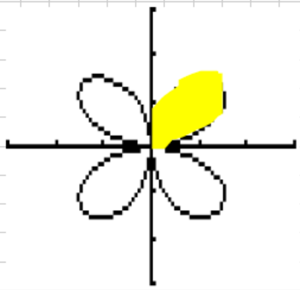
$\left.\frac{dy}{dx}\right|_{\theta=0} = 0 \Rightarrow$ Tangent: $y=0$

$\left.\frac{dy}{dx}\right|_{\theta=\pi} = 0 \Rightarrow$ Tangent: $y=0$

At $\theta = \pi/2$ and $\theta = 3\pi/2$ $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0 \Rightarrow$ m is undefined

$\boxed{\text{Tangent: } x = 0}$

2. One petal of $r = 2\sin 2\theta$



$\sin 2\theta = 0$

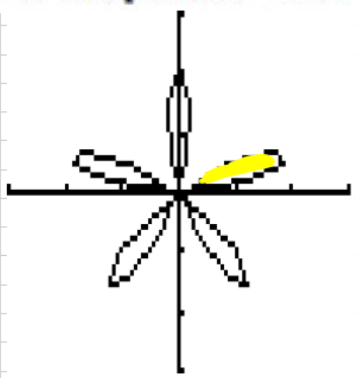
$2\theta = n\pi$

$\theta = \frac{n\pi}{2}$

$$A = \int_0^{\pi/2} \frac{1}{2}(2\sin(2\theta))^2 d\theta = 2\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

$$\boxed{A \approx 6.283}$$

3. One petal of $r = \sin 5\theta$



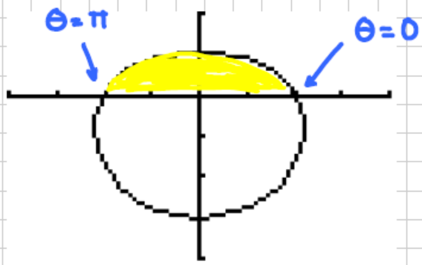
$\sin(5\theta) = 0$

$5\theta = 0 \longrightarrow \theta = 0$

$5\theta = \pi \longrightarrow \theta = \pi/5$

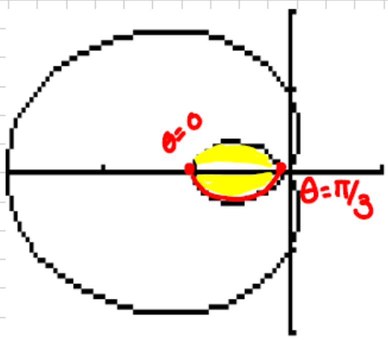
$$A = \int_0^{\pi/5} \frac{1}{2}(\sin(5\theta))^2 d\theta \approx \boxed{0.157}$$

4. Interior of $r = 2 - \sin\theta$ (above polar axis)




$$A = \frac{1}{2}\int_0^{\pi} (2-\sin\theta)^2\,d\theta \approx \boxed{3.069}$$

5. Inner loop of $r = 1 - 2\cos\theta$

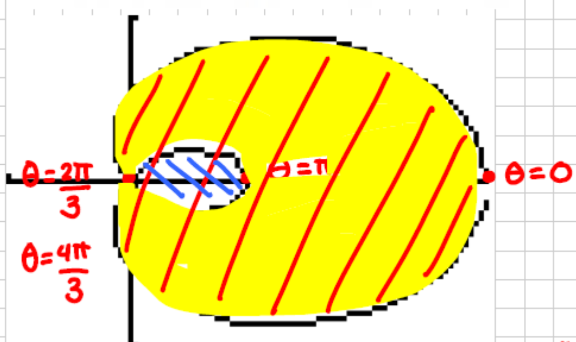



| $\theta$ | $r$ |
|---|---|
| $0$ | $-1$ |
| $\pi/3$ | $0$ |

Multiply by 2 to take advantage of the symmetry!

$$A = \int_0^{\pi/3} (1-2\cos\theta)^2\,d\theta \approx \boxed{0.544}$$

6. Between the loops of $r = 1 + 2\cos\theta$

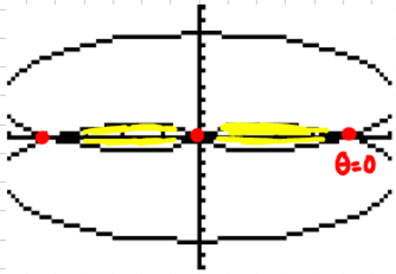



$$\int_0^{2\pi/3} \frac{1}{2}(1+2\cos\theta)^2\,d\theta = 9.4248$$

$$\int_{2\pi/3}^{4\pi/3} \frac{1}{2}(1+2\cos\theta)^2\,d\theta = 0.5435$$

$$\boxed{A \approx 8.881}$$

11. Common interior of $r_1 = 4(1+\sin\theta)$ and $r_2 = 4(1-\sin\theta)$




| $\theta$ | $r_1$ | $r_2$ |
|---|---|---|
| 0 | 4 | 4 |
| $\pi/2$ | 8 | 0 |