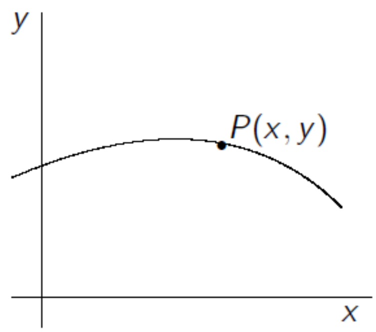
MAT187H1F Lec0101 Burbulla

Chapter 11 Lecture Notes

Spring 2017

What Is A Parametric Curve?



1. Let a point P on a curve have Cartesian coordinates (x, y)
2. We can think of the curve as being traced out as the point P moves along it.
3. In this way we can think of both x and y as functions of t.
4. $x = f(t)$ and $y = g(t)$ are called parametric equations of the curve; t is called a parameter.

Example 1

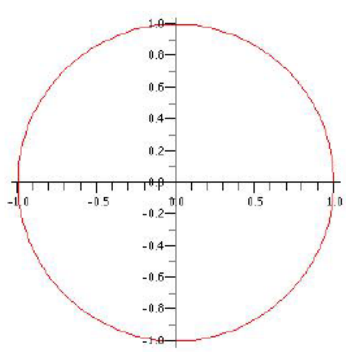


Figure: The circle $x^2 + y^2 = 1$

Let

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

with $0 \leq t \leq 2\pi$. These are parametric equations of a circle:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

In terms of t, the circle is being traced out counter clockwise.

Example 2

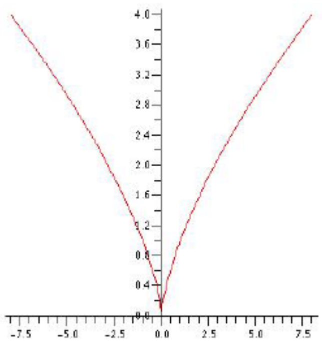


Figure: The curve $y^3 = x^2$

Let

$$\begin{cases} x &= t^3 \\ y &= t^2 \end{cases}$$

with $t \in \mathbb{R}$. We can eliminate the parameter:

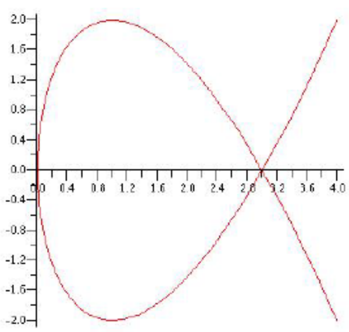
$$t = x^{1/3} \Rightarrow y = \left(x^{1/3}\right)^2 = x^{2/3}.$$

We could also say

$$y^3 = t^6 = x^2 \Rightarrow y^3 = x^2.$$

Observe: $t = 1 \Rightarrow x = 1, y = 1$; $t = 2 \Rightarrow x = 8, y = 4$; etc.

Example 3



Let

$$\begin{cases} x &= t^2 \\ y &= t^3 - 3t \end{cases}$$

with $t \in \mathbb{R}$. For this example, you could eliminate the parameter, but it would be messy. For instance, you would have to consider two cases:

$$t = \pm\sqrt{x}.$$

Generally, parametric curves are most suitable for relations in which no single function can easily describe the curve.

Most General Form of a Curve

Parametric curves are the most general type of curve. The function $y = f(x)$ can be described parametrically by

$$\begin{cases} x = t \\ y = f(t) \end{cases}, t \text{ a parameter}$$

In essence, x is the parameter of a curve described by the function $y = f(x)$. All curves we shall look at later, polar curves and curves in 3 dimensions, can be described parametrically.

Chapter 11: Parametric and Polar Curves
11.1 Parametric Equations
11.2 Polar Coordinates
11.3 Calculus in Polar Coordinates

Derivatives of Parametric Curves

To do calculus with parametric curves we will need formulas for derivatives in terms of the parameter. Suppose x and y are functions of a parameter t. Then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

As always, the derivative will be the slope of the curve at the point (x, y); the difference is that everything will be calculated in terms of the parameter t.

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Example 4; Example 2 Revisited

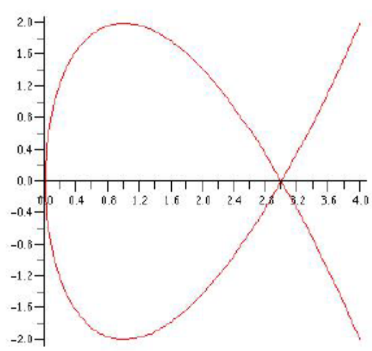
We have $x = t^3, y = t^2$; so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}, t \neq 0.$$

This is the same answer you get if you eliminate the parameter:

$$t = x^{1/3} \Rightarrow y = x^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-1/3}.$$

Example 5; Example 3 Revisited



We have $x = t^2, y = t^3 - 3t$; so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}.$$

The critical points are at

$$t = \pm 1; t = 0.$$

The coordinates of the critical points are $(0, 0), (1, \pm 2)$.

The Second Derivative in Parametric Form

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}, \text{ with } y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

From Examples 4 and 2: $x = t^3, y = t^2$; $\frac{dy}{dx} = \frac{2}{3t}$. So

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2}{3t}\right)}{\frac{dt^3}{dt}} = \frac{-\frac{2}{3t^2}}{3t^2} = -\frac{2}{9t^4}.$$

Or, since $t = x^{1/3}$ and $y = x^{2/3}$: $\frac{dy}{dx} = \frac{2}{3x^{1/3}} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{9x^{4/3}}$.

Example 6; Examples 3 and 5 Revisited

In Examples 3 and 5 we had: $x = t^2, y = t^3 - 3t$, and

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{3t}{2} - \frac{3}{2t}.$$

Hence

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t}{2} - \frac{3}{2t}\right)}{\frac{dt^2}{dt}} = \frac{\left(\frac{3}{2} + \frac{3}{2t^2}\right)}{2t} = \frac{3}{4}\frac{t^2+1}{t^3}.$$

Thus

$$\frac{d^2y}{dx^2} > 0 \Leftrightarrow t > 0 \text{ and } \frac{d^2y}{dx^2} < 0 \Leftrightarrow t < 0.$$

So $(0, 0)$ is an inflection point. (Check the graph!)

Example 7: The Cycloid

Consider the parametric curve with $x = t - \sin t$; $y = 1 - \cos t$. We shall do a typical analysis of the first two derivatives to figure out how the graph of this curve, which is called a cycloid, looks:

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t} \text{ and } \frac{d^2y}{dx^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^3} = -\frac{1}{(1 - \cos t)^2},$$

as you may check. Since the second derivative is non-positive, the graph will always be concave down. There are two types of critical points:

1. vertical tangents, when $\cos t = 1$: namely at $t = 0, \pm 2\pi, \pm 4\pi, \ldots$. So $(x, y) = (2k\pi, 0), k \in \mathbb{Z}$.
2. horizontal tangents, when $\sin t = 0$ but $\cos t \neq 1$; namely at $t = \pm\pi, \pm 3\pi, \ldots$. So $(x, y) = ((2k+1)\pi, 2), k \in \mathbb{Z}$.

Graph of the Cycloid, for $0 \leq t \leq 4\pi$

1. The graph is increasing if

$$\frac{dy}{dx} > 0$$
$$\Leftrightarrow \quad \sin t > 0$$
$$\Leftrightarrow \quad 0 < t < \pi, 2\pi < t < 3\pi$$

2. The graph is decreasing if

$$\frac{dy}{dx} < 0$$
$$\Leftrightarrow \quad \sin t < 0$$
$$\Leftrightarrow \quad \pi < t < 2\pi, 3\pi < t < 4\pi$$

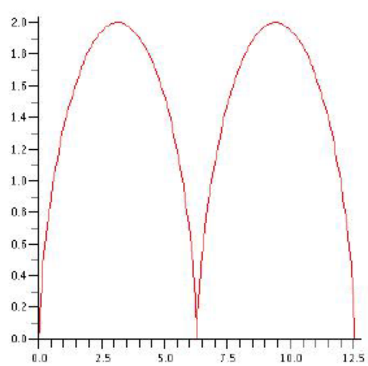


Figure: A cycloid.

Integral Formulas in Parametric Form

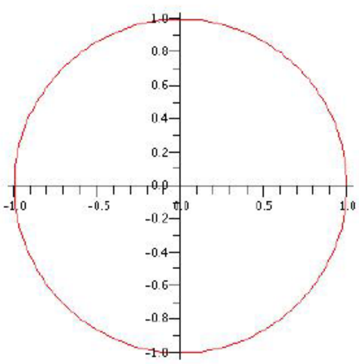
Every integral formula that you are familiar with from Chapter 6:

1. area, eg. $A = \int_a^b y\,dx$
2. volume, eg. $V = \int_a^b \pi y^2\,dx$ or $V = \int_a^b 2\pi x\,y\,dx$
3. length, eg. $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
4. surface area, eg. $SA = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$;

can be applied to parametric curves by substituting for x and y in terms of the parameter t, and then simplifying. In other words, these problems are just fancy change of variable problems. But, you must be careful in changing the limits of integration as you make the substitutions. Your new integral will be in terms of t, and so should the limits of integration.

Example 8: Area of a Circle

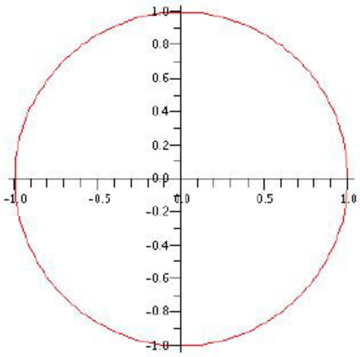
Circle of radius a has parametric equations $x = a\cos t$, $y = a\sin t$.



$$\begin{aligned} A &= 4\int_0^a y\,dx \\ &= 4\int_{\pi/2}^0 a\sin t \cdot (-a\sin t)\,dt \\ &= 4a^2 \int_0^{\pi/2} \sin^2 t\,dt \\ &= 4a^2 \int_0^{\pi/2} \left(\frac{1-\cos(2t)}{2}\right) dt \\ &= 2a^2 \left[t - \frac{\sin(2t)}{2}\right]_0^{\pi/2} = \pi a^2 \end{aligned}$$

Example 9: Volume of a Sphere

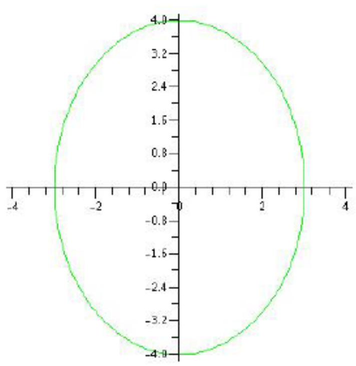
Rotate the quarter circle around the x-axis, and multiply by two.



$$\begin{aligned} V &= 2\int_0^a \pi y^2\,dx \\ &= 2\pi\int_{\pi/2}^0 a^2\sin^2 t\cdot(-a\sin t)\,dt \\ &= 2\pi a^3\int_{\pi/2}^0 (1-\cos^2 t)(-\sin t)\,dt \\ &= 2\pi a^3\int_0^1 (1-u^2)\,du,\ \text{with } u=\cos t \\ &= 2\pi a^3\left[u-\frac{1}{3}u^3\right]_0^1 = \frac{4}{3}\pi a^3 \end{aligned}$$

Example 10: Area of the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

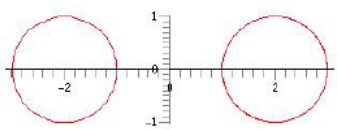
Parametric equations of an ellipse: $x = a\cos t, y = b\sin t$.



$$\begin{aligned} A &= 4\int_0^a y\,dx \\ &= 4\int_{\pi/2}^0 b\sin t\cdot(-a\sin t)\,dt \\ &= 4ab\int_0^{\pi/2}\sin^2 t\,dt \\ &= 4ab\int_0^{\pi/2}\left(\frac{1-\cos(2t)}{2}\right)dt \\ &= 2ab\left[t-\frac{\sin(2t)}{2}\right]_0^{\pi/2} = \pi ab \end{aligned}$$

Example 11: Volume of a Torus

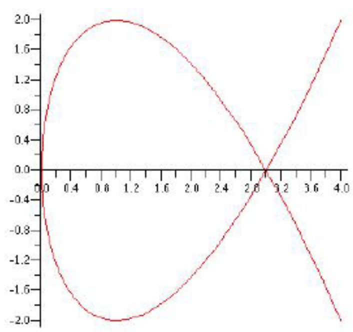
A torus is obtained by rotating a circle, $x = b + a\cos t$, $y = a\sin t$, for $a < b$, around the y-axis.



$$\begin{aligned} V &= 2\int_{b-a}^{b+a} 2\pi x\, y\, dx \\ &= 4\pi \int_{\pi}^{0} (b + a\cos t)\cdot a\sin t\,(-a\sin t)\, dt \\ &= 4\pi a^2 \int_0^{\pi} (b\sin^2 t + a\cos t\sin^2 t)\, dt \\ &= 4\pi a^2 b \int_0^{\pi} \left(\frac{1-\cos(2t)}{2}\right) dt + 0 \\ &= 2\pi a^2 b \left[t - \frac{\sin(2t)}{2}\right]_0^{\pi} = 2\pi^2 a^2 b \end{aligned}$$

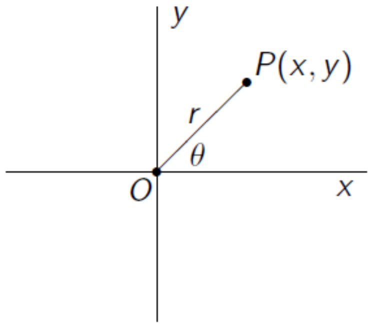
Example 12

Find the area within the loop of the curve $x = t^2, y = t^3 - 3t$.



$$\begin{aligned} A &= 2\int_0^3 y\, dx, \text{ if } y > 0 \\ &= 2\int_0^{-\sqrt{3}} (t^3 - 3t)\, 2t\, dt \\ &= 2\int_{-\sqrt{3}}^{0} (6t^2 - 2t^4)\, dt \\ &= 2\left[2t^3 - \frac{2}{5}t^5\right]_{-\sqrt{3}}^{0} = \frac{24}{5}\sqrt{3} \end{aligned}$$

What Are Polar Coordinates?



1. Let the point P have Cartesian coordinates (x, y)
2. Let r be the length of OP.
3. Let θ be the angle the line OP makes with the positive x-axis.
4. Then r and θ are called the polar coordinates of P.

We have

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Equivalently,

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

The value of θ depends on which quadrant (x, y) is in.

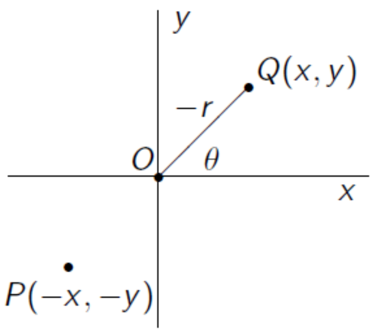
Example 1

Find the polar coordinates of the following points. Warning: polar coordinates are not unique because θ is not unique.

(x, y)	(r, θ) with angles chosen from $[-2\pi, 2\pi]$
$(2, 0)$	$r = 2, \theta = 0$ or $\theta = 2\pi$
$(1, 1)$	$r = \sqrt{2}, \theta = \pi/4$
$(0, 1)$	$r = 1, \theta = \pi/2$
$(0, -3)$	$r = 3, \theta = -\pi/2$ or $\theta = 3\pi/2$
$(-2, 0)$	$r = 2, \theta = \pi$ or $\theta = -\pi$
$(-1, -\sqrt{3})$	$r = 2, \theta = 4\pi/3$ or $\theta = -2\pi/3$
$(3, -4)$	$r = \sqrt{9+16} = 5, \theta = \tan^{-1}\left(-\frac{4}{3}\right) \simeq -0.927$ radians
$(0, 0)$	$r = 0$, θ can be anything.

Negative Values of r

According to the definition of polar coordinates, r should be positive. However, we can extend the definition to include negative values of r. Suppose $r < 0$.



1. Let Q have coordinates (x, y) and polar coordinates $(-r, \theta)$.
2. Then the point P with polar coordinates (r, θ) is defined to be the point with polar coordinates $(-r, \theta \pm \pi)$.
3. That is, $\overrightarrow{OP} = -\overrightarrow{OQ}$.

Example 2

The point with polar coordinates

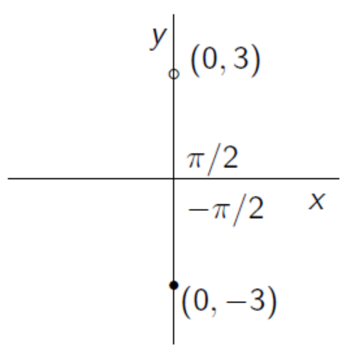
$$(r, \theta) = \left(-3, \frac{\pi}{2}\right)$$

is the same as the point with polar coordinates

$$\left(3, \frac{\pi}{2} - \pi\right) = \left(3, -\frac{\pi}{2}\right).$$

Its Cartesian coordinates are

$$(x, y) = (0, -3).$$



Polar Curves

If $r = f(\theta)$, the polar curve of f is the set of all points whose polar coordinates (r, θ) satisfy the equation

$$r = f(\theta).$$

There is a difference between the graph of $r = f(\theta)$ and the polar curve of the equation $r = f(\theta)$. The former is the set of points (x, y) such that

$$x = \theta \text{ and } y = f(\theta).$$

These graphs are nothing new; exactly like all the previous examples you've done in Chapters 3 and 4. The polar curve is something new. On the polar curve, the coordinates are $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$.

Example 3: $r = 1$

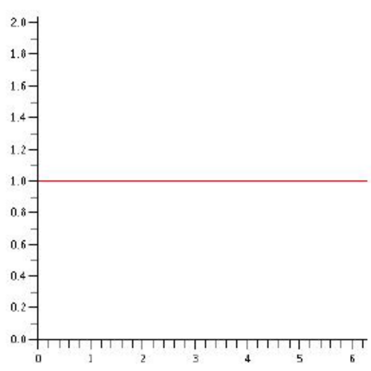


Figure: Graph of $r = 1$

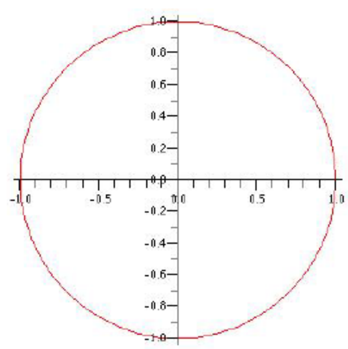


Figure: Polar curve: $r = 1$

Example 4: $r = 2\sin\theta$

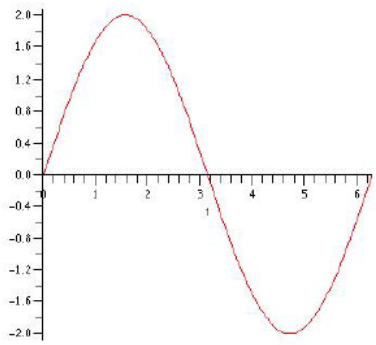


Figure: Graph of $r = 2\sin\theta$

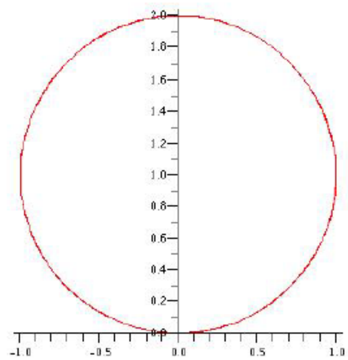


Figure: Polar curve: $r = 2\sin\theta$

Example 4, Continued

You can convince yourself that the polar curve is a circle by changing the polar equation into a Cartesian equation:

$$\begin{aligned} r = 2\sin\theta \quad &\Rightarrow \quad r^2 = 2r\sin\theta \\ &\Rightarrow \quad x^2 + y^2 = 2y \\ &\Rightarrow \quad x^2 + y^2 - 2y = 0 \\ &\Rightarrow \quad x^2 + y^2 - 2y + 1 = 1 \\ &\Rightarrow \quad x^2 + (y-1)^2 = 1 \end{aligned}$$

This last equation is the equation of a circle with centre $(0, 1)$ and radius 1. Warning: it won't always be this easy to switch from the equation of a polar curve to its Cartesian equation! In fact, we will rarely do it.

Exampl 5: $r = 1 + \cos\theta$

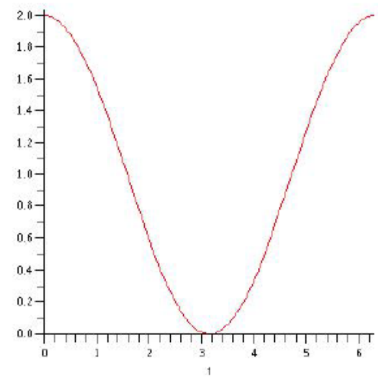


Figure: Graph of the function $r = 1 + \cos\theta$

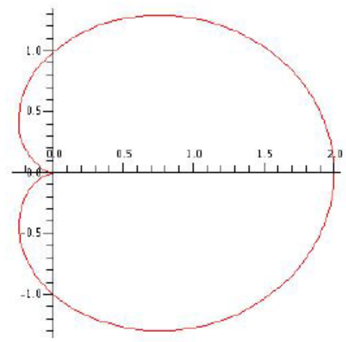


Figure: Cardioid with polar equation: $r = 1 + \cos\theta$

Example 6: $r = 2 + \cos\theta$

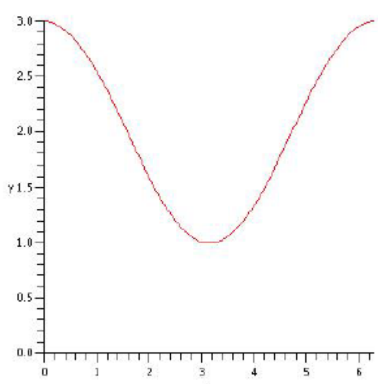


Figure: Graph of the function $r = 2 + \cos\theta$

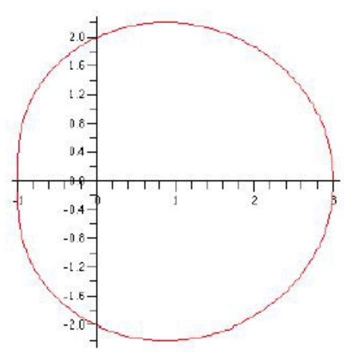


Figure: Curve with polar equation: $r = 2 + \cos\theta$

Example 7: $r = 1 - 2\cos\theta$

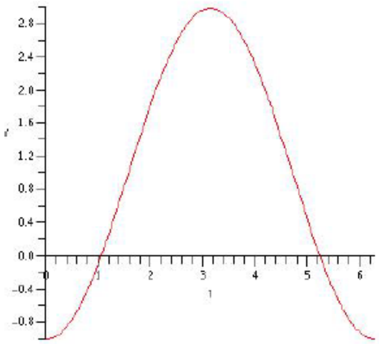


Figure: Graph of the function $r = 1 - 2\cos\theta$

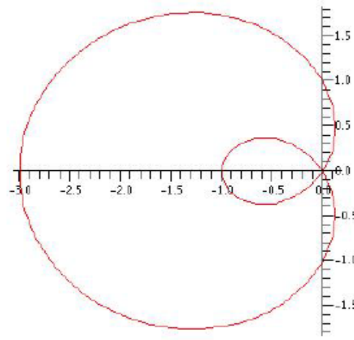


Figure: Limaçon with polar equation: $r = 1 - 2\cos\theta$

Example 7, Continued

The limaçon has an interesting feature: a loop within a loop. For which angles does the curve of $r = 1 - 2\cos\theta$ trace out the outer loop? the inner loop? The two loops have only one point in common, the origin $(0, 0)$, at which $r = 0$. The boundary angles of the loops are found by solving

$$r = 0 \Leftrightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}.$$

The outer loop, when $r > 0$, is traced for

$$\frac{\pi}{3} < \theta < \frac{5\pi}{3}.$$

For $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$, $r < 0$ and the curve traces out the inner loop.

Example 8: $r = 2\sin(2\theta)$

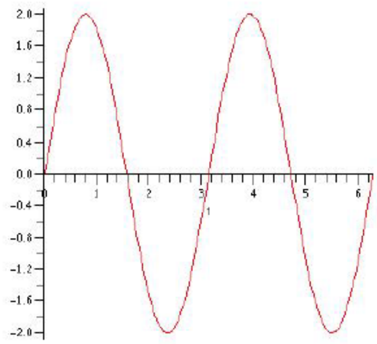


Figure: Graph of the function $r = 2\sin(2\theta)$

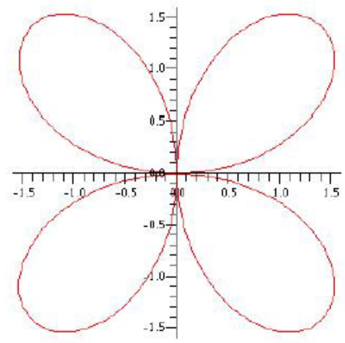


Figure: Four leaved rose with polar equation: $r = 2\sin(2\theta)$

Example 9: $r = 2\sin(3\theta)$

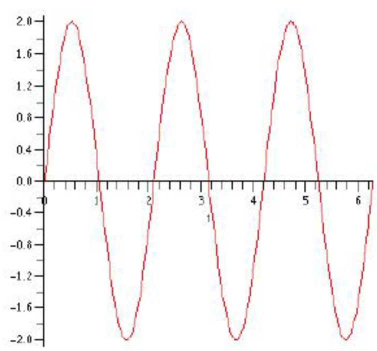


Figure: Graph of the function $r = 2\sin(3\theta)$

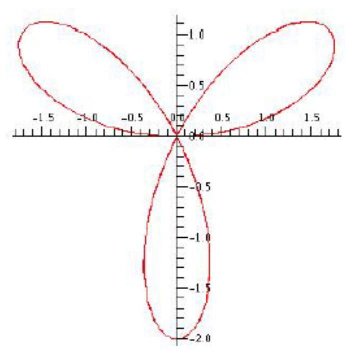


Figure: Three leaved rose with polar equation: $r = 2\sin(3\theta)$

Example 10: $r = \sin\theta + \cos\theta$

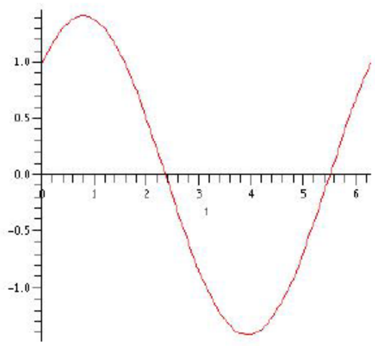


Figure: Graph of the function $r = \sin\theta + \cos\theta$

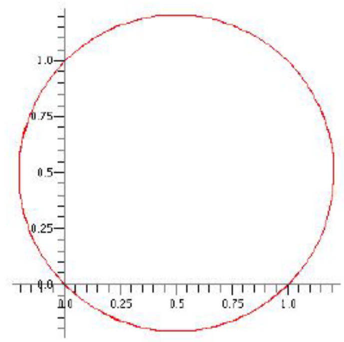


Figure: Circle with polar equation: $r = \sin\theta + \cos\theta$

Example 10, Continued

We can confirm that the polar curve of $r = \sin\theta + \cos\theta$ is a circle, by finding its Cartesian equation.

$$\begin{aligned} r = \sin\theta + \cos\theta \quad &\Rightarrow \quad r^2 = r\sin\theta + r\cos\theta \\ &\Rightarrow \quad x^2 + y^2 = y + x \\ &\Rightarrow \quad x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} \\ &\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2} \end{aligned}$$

So the circle has centre

$$\left(\frac{1}{2}, \frac{1}{2}\right)$$

and radius $1/\sqrt{2}$.

Example 11: $r = \theta/25$

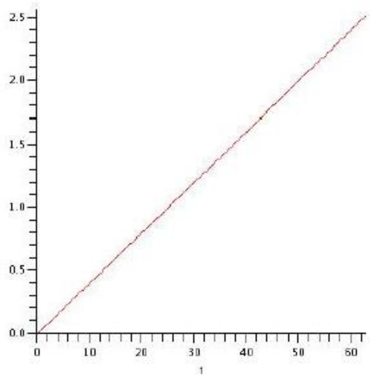


Figure: Graph of the function $r = \theta/25, \theta \geq 0$

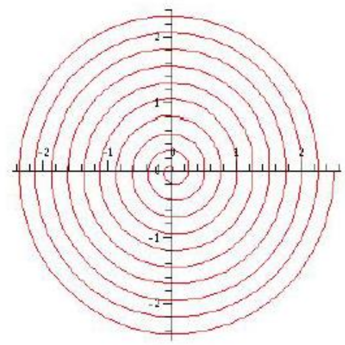


Figure: Archimedean spiral with polar equation: $r = \theta/25, \theta \geq 0$

Example 12: $r = e^{\theta/25}$

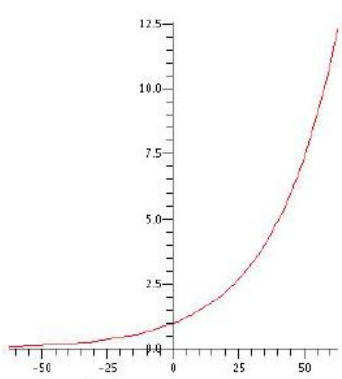


Figure: Graph of the function $r = e^{\theta/25}$

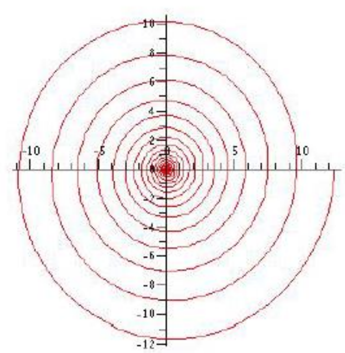


Figure: Logarithmic spiral with polar equation: $r = e^{\theta/25}$

Derivatives for Polar Curves

Since the parametric equations for a polar curve are

$$x = r\cos\theta, y = r\sin\theta,$$

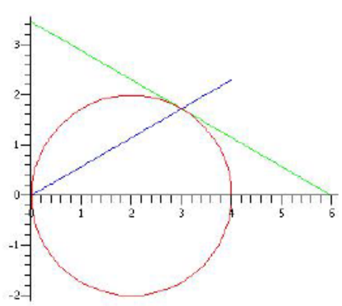
the first derivative of a polar curve is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}.$$

The formula for the second derivative is so messy we won't even try to write it down!

Example 1

Find the slope of the tangent line to the circle with polar equation $r = 4\cos\theta$ at the point with $\theta = \pi/6$.



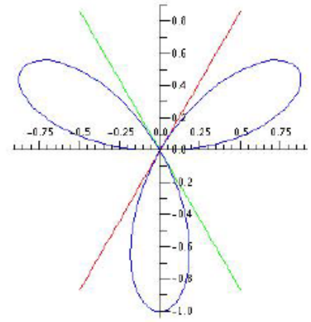
$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta} \\ &= \frac{-4\sin^2\theta + 4\cos^2\theta}{-4\cos\theta\sin\theta - 4\cos\theta\sin\theta} \\ &= \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta} = -\cot(2\theta) \end{aligned}$$

So at $\theta = \pi/6$ the slope is $-\cot(\pi/3) = -1/\sqrt{3}$.

Example 2: Tangent Lines to Polar Curves at the Origin

If $r = 0$ for $\theta = \alpha$, and $dr/d\theta \neq 0$ at $\theta = \alpha$, then

$$\left.\frac{dy}{dx}\right|_{\theta=\alpha} = \frac{\sin\alpha \left.\dfrac{dr}{d\theta}\right|_{\theta=\alpha} + 0}{\cos\alpha \left.\dfrac{dr}{d\theta}\right|_{\theta=\alpha} - 0} = \tan\alpha.$$

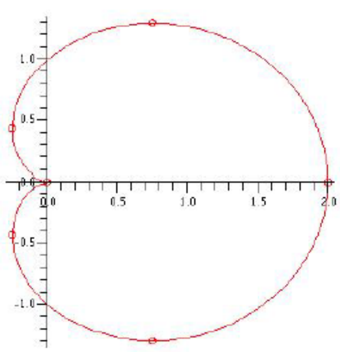


That is, the line $\theta = \alpha$ is tangent to the curve at the origin.

The figure to the right illustrates the example $r = \sin 3\theta$ for which the tangent lines at the origin have equations $\theta = 0, \pi/3$ or $2\pi/3$.

Example 3: Critical Points of a Cardioid

Looking at the graph of the cardioid $r = 1 + \cos\theta$, we expect to find six critical points: namely where the graph has its three vertical tangents and its three horizontal tangents.



The parametric equations of the cardioid are

$$x = r\cos\theta;\, y = r\sin\theta$$

so

$$\begin{cases} x = (1+\cos\theta)\cos\theta \\ y = (1+\cos\theta)\sin\theta \end{cases}$$

Horizontal Tangents to the Cardioid $r = 1 + \cos\theta$

$$\begin{aligned}
\frac{dy}{d\theta} = 0 \quad &\Leftrightarrow \quad \frac{d}{d\theta}\left((1+\cos\theta)\sin\theta\right) = 0 \\
&\Leftrightarrow \quad -\sin^2\theta + \cos\theta + \cos^2\theta = 0 \\
&\Leftrightarrow \quad \cos^2\theta + \cos\theta - (1-\cos^2\theta) = 0 \\
&\Leftrightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0 \\
&\Leftrightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0 \\
&\Leftrightarrow \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1
\end{aligned}$$

Find the critical points: $\cos\theta = -1 \Rightarrow r = 0 \Rightarrow (x, y) = (0, 0)$.

$$\cos\theta = \frac{1}{2} \Rightarrow r = \frac{3}{2} \text{ and } \sin\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow (x, y) = \left(\frac{3}{4}, \pm\frac{3\sqrt{3}}{4}\right).$$

Vertical Tangents to the Cardioid $r = 1 + \cos\theta$

$$\begin{aligned}
\frac{dx}{d\theta} = 0 \quad &\Leftrightarrow \quad \frac{d}{d\theta}\left((1+\cos\theta)\cos\theta\right) = 0 \\
&\Leftrightarrow \quad -\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta = 0 \\
&\Leftrightarrow \quad \sin\theta(1 + 2\cos\theta) = 0 \\
&\Leftrightarrow \quad \cos\theta = -\frac{1}{2} \text{ or } \sin\theta = 0
\end{aligned}$$

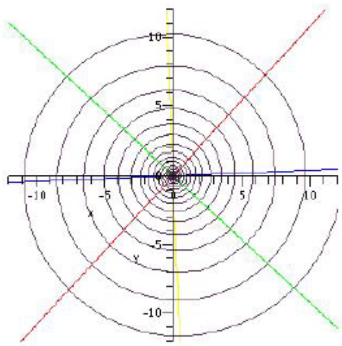
Find the critical points:

$$\cos\theta = -\frac{1}{2} \Rightarrow r = \frac{1}{2} \text{ and } \sin\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow (x, y) = \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right).$$

$\sin\theta = 0 \Rightarrow \cos\theta = \pm 1 \Rightarrow r = 0$ or 2. Take $(x, y) = (2, 0)$.

Example 4: Logarithmic Spiral $r = e^{\theta/25}$ Revisited

Note: for the following, the algebra is easier if $r = e^{\theta}$, but then the curve spirals out so quickly that it is hard to get a suitable graph.



The parametric equations of this spiral are

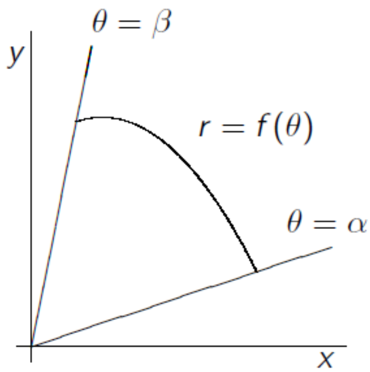
$$x = e^{\theta/25}\cos\theta, y = e^{\theta/25}\sin\theta;$$

$$\frac{dy}{dx} = \frac{1/25e^{\theta/25}\sin\theta + e^{\theta/25}\cos\theta}{1/25e^{\theta/25}\cos\theta - e^{\theta/25}\sin\theta} = \frac{\sin\theta + 25\cos\theta}{\cos\theta - 25\sin\theta}$$

$$\begin{aligned} \frac{dy}{dx} = m \quad &\Leftrightarrow \quad \frac{\sin\theta + 25\cos\theta}{\cos\theta - 25\sin\theta} = m \\ &\Leftrightarrow \quad \sin\theta + 25\cos\theta = m(\cos\theta - 25\sin\theta) \\ &\Leftrightarrow \quad (1 + 25m)\sin\theta = (m - 25)\cos\theta \\ &\Leftrightarrow \quad \tan\theta = \frac{m - 25}{1 + 25m} \end{aligned}$$

1. $m = 0 \Rightarrow \tan\theta = -25$. This means that all the points on the spiral with slope zero lineup along the line $y = -25x$.
2. $m = \pm\infty \Rightarrow \tan\theta = \frac{1}{25}$. This means that all the points on the spiral with undefined slope lineup along the line $y = x/25$.
3. $m = 1 \Rightarrow \tan\theta = -\frac{12}{13}$. This means that all the points on the spiral with slope 1 lineup along the line $y = -\frac{12}{13}x$.
4. $m = -1 \Rightarrow \tan\theta = \frac{13}{12}$. This means that all the points on the spiral with slope -1 lineup along the line $y = \frac{13}{12}x$.

Area within a Polar Curve



The area within the polar curve with equation $r = f(\theta)$, between the rays $\theta = \alpha$ and $\theta = \beta$, is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Proof: see the book. It's not hard: use partitions of the interval $[\alpha, \beta]$ and Riemann sums to approximate the area.

You need to know that the area of a circle sector is given by $\frac{1}{2}r^2\theta$.

Example 5: Area In A Cardioid

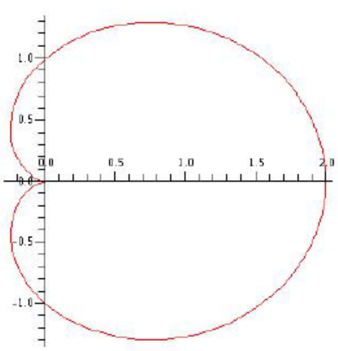


Figure: Cardioid with polar equation: $r = 1 + \cos\theta$

By symmetry, we can double the area of the top half. So A

$$= 2\left(\frac{1}{2}\int_0^{\pi} r^2\, d\theta\right)$$

$$= \int_0^{\pi} (1+\cos\theta)^2\, d\theta$$

$$= \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin(2\theta)}{4}\right]_0^{\pi} = \frac{3\pi}{2}$$

Example 6

Fiind the area of one petal of the three-leaved rose $r = 2\sin(3\theta)$.

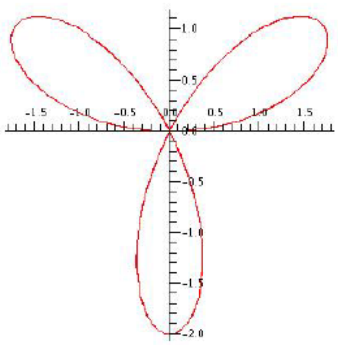


Figure: $r = 2\sin(3\theta)$

$$r = 0 \;\Rightarrow\; 3\theta = 0 \text{ or } \pi \;\Rightarrow\; \theta = 0 \text{ or } \frac{\pi}{3}.$$

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/3} 4\sin^2(3\theta)\,d\theta \\ &= \int_0^{\pi/3} (1 - \cos(6\theta))\,d\theta \\ &= \left[\theta - \frac{1}{6}\sin(6\theta)\right]_0^{\pi/3} = \frac{\pi}{3} \end{aligned}$$

Chapter 11: Parametric and Polar Curves
11.1 Parametric Equations
11.2 Polar Coordinates
11.3 Calculus in Polar Coordinates

Example 7

Find the area inside the circle with polar equation $r = \sin\theta + \cos\theta$.

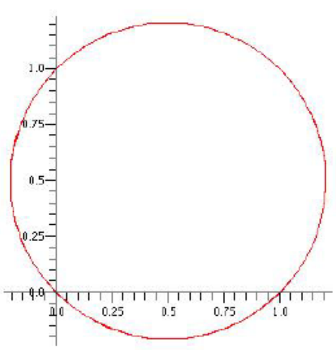


Figure: $r = \sin\theta + \cos\theta$

$$\begin{aligned} r = 0 &\Rightarrow \sin\theta + \cos\theta = 0 \\ &\Rightarrow \tan\theta = -1 \\ &\Rightarrow \theta = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4} \\ \Rightarrow A &= \frac{1}{2}\int_{-\pi/4}^{3\pi/4} (\sin\theta + \cos\theta)^2\,d\theta \\ &= \frac{1}{2}\int_{-\pi/4}^{3\pi/4} (1 + 2\sin\theta\cos\theta)\,d\theta \\ &= \frac{1}{2}\left[\theta + \sin^2\theta\right]_{-\pi/4}^{3\pi/4} = \frac{\pi}{2} \end{aligned}$$

Example 8

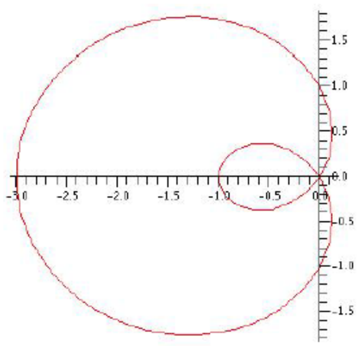


Figure: $r = 1 - 2\cos\theta$

Find the area within each of the inner and outer loops of the limaçon: Recall, that the outer loop, $r > 0$, is traced out for

$$\theta \in [\pi/3, 5\pi/3],$$

and that inner loop, $r < 0$, is traced out for

$$\theta \in [-\pi/3, \pi/3].$$

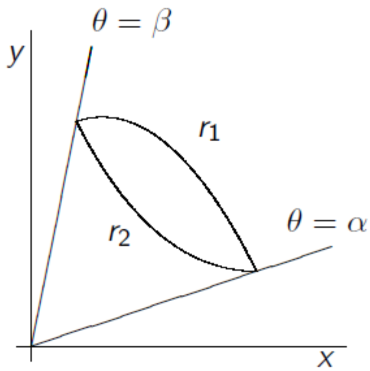
Let A_i be the area of the inner loop; let A_o be the area of the outer loop.

Example 8, Continued

$$\begin{aligned}
A_0 &= 2\left(\frac{1}{2}\int_{\pi/3}^{\pi}(1-2\cos\theta)^2\,d\theta\right) = \int_{\pi/3}^{\pi}(1-4\cos\theta+4\cos^2\theta)\,d\theta \\
&= \int_{\pi/3}^{\pi}(3-4\cos\theta+2\cos(2\theta))\,d\theta = [3\theta-4\sin\theta+\sin(2\theta)]_{\pi/3}^{\pi} \\
&= 3\pi - 3(\pi/3) + 4\sin(\pi/3) - \sin(2\pi/3) = 2\pi + \frac{3}{2}\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
A_i &= 2\left(\frac{1}{2}\int_{0}^{\pi/3}(1-2\cos\theta)^2\,d\theta\right) = [3\theta-4\sin\theta+\sin(2\theta)]_{0}^{\pi/3} \\
&= 3(\pi/3) - 4\sin(\pi/3) + \sin(2\pi/3) = \pi - \frac{3}{2}\sqrt{3}
\end{aligned}$$

Area Between Curves



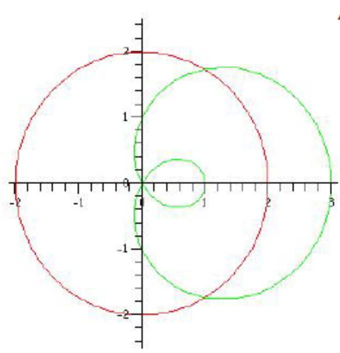
If the two polar curves r_1 and r_2 intersect at $\theta = \alpha$ and $\theta = \beta$, then the area between the curves is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right)\, d\theta.$$

This is simply the area within r_1 minus the area within r_2.

Example 9

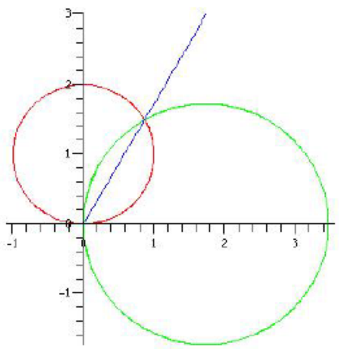
Find the area inside the limaçon $r = 1 + 2\cos\theta$ but outside the circle $r = 2$. Have: $1 + 2\cos\theta = 2 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pm\pi/3$.



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\pi/3} \left((1+2\cos\theta)^2 - 2^2\right)\, d\theta\right) \\ &= \int_0^{\pi/3} \left(-3 + 4\cos\theta + 4\cos^2\theta\right)\, d\theta \\ &= \int_0^{\pi/3} \left(-1 + 4\cos\theta + 2\cos(2\theta)\right)\, d\theta \\ &= \left[-\theta + 4\sin\theta + \sin(2\theta)\right]_0^{\pi/3} \\ &= -\pi/3 + 4\sin(\pi/3) + \sin(2\pi/3) \\ &= 5\sqrt{3}/2 - \pi/3 \end{aligned}$$

Example 10: A Very Tricky Example

Find the area inside the circle $r = 2\sqrt{3}\cos\theta$ and inside the circle $r = 2\sin\theta$. Have: $2\sqrt{3}\cos\theta = 2\sin\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$.



There is another intersection point, at the origin. However, the two circles pass through the origin for different values of θ. For the circle $r = 2\sin\theta$, $r = 0$ if $\theta = 0$. But for the circle $r = 2\sqrt{3}\cos\theta$, $r = 0$ if $\theta = \pi/2$. This means that it will require two separate integrals to find the total area of the region common to these two circles.

Example 10, Continued

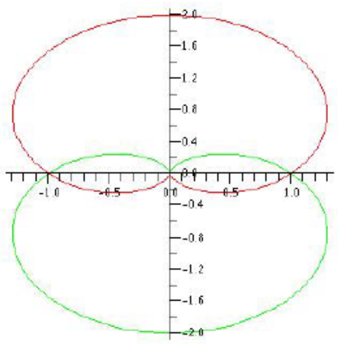
Let A_1 be the area inside $r = 2\sin\theta$ for $\theta \in [0, \pi/3]$; let A_2 be the area inside $r = 2\sqrt{3}\cos\theta$ for $\theta \in [\pi/3, \pi/2]$. $A = A_1 + A_2$.

$$A_1 = \frac{1}{2}\int_0^{\pi/3} 4\sin^2\theta\, d\theta = \int_0^{\pi/3} (1 - \cos(2\theta))\, d\theta = \left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

$$A_2 = \frac{1}{2}\int_{\pi/3}^{\pi/2} 12\cos^2\theta\, d\theta = 3\int_{\pi/3}^{\pi/2} (1 + \cos(2\theta))\, d\theta = 3\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\pi/3}^{\pi/2} = \frac{\pi}{2} - \frac{3}{4}\sqrt{3}$$

Example 11

Consider the two cardioids $r = 1 \pm \sin\theta$.



$$\begin{aligned} 1 + \sin\theta = 1 - \sin\theta \quad &\Rightarrow \quad 2\sin\theta = 0 \\ &\Rightarrow \quad \theta = 0 \text{ or } \pi \end{aligned}$$

The third intersection point is the origin: $r = 1 + \sin\theta = 0$ if $\theta = -\pi/2$; $r = 1 - \sin\theta = 0$ if $\theta = \pi/2$. So again, finding areas of regions determined by these two cardioids can be quite tricky.

Example 11, Continued

The area inside $r = 1 + \sin\theta$ but outside $r = 1 - \sin\theta$ is given by

$$\begin{aligned} \frac{1}{2}\int_0^{\pi} \left((1+\sin\theta)^2 - (1-\sin\theta)^2\right)\, d\theta &= \frac{1}{2}\int_0^{\pi} 4\sin\theta\, d\theta \\ &= \left[-2\cos\theta\right]_0^{\pi} = 4 \end{aligned}$$

The area of the region common to both cardioids is given by

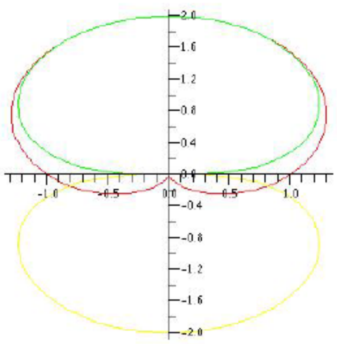
$$4\left(\frac{1}{2}\int_0^{\pi/2}(1-\sin\theta)^2\, d\theta\right) = 2\int_0^{\pi/2}(1 - 2\sin\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{\pi/2}(3 - 4\sin\theta - \cos(2\theta))\, d\theta = \left[3\theta + 4\cos\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$$

$$= \frac{3\pi}{2} - 4$$

Example 12: The Problem of Intersection Points

Consider the polar curves $r = 1 + \sin\theta$ and $r^2 = 4\sin\theta$. The polar curve $r^2 = 4\sin\theta$ comes in two parts: $r = \pm 2\sqrt{\sin\theta}$, for $\theta \in [0, \pi]$.



From the graphs we expect four intersection points. Substituting for r from one equation into the other:

$$\begin{aligned} & (1+\sin\theta)^2 = 4\sin\theta \\ \Rightarrow\ & 1 + 2\sin\theta + \sin^2\theta = 4\sin\theta \\ \Rightarrow\ & 1 - 2\sin\theta + \sin^2\theta = 0 \\ \Rightarrow\ & (1-\sin\theta)^2 = 0 \\ \Rightarrow\ & \sin\theta = 1 \\ \Rightarrow\ & \theta = \pi/2, \text{ at } (x, y) = (0, 2). \end{aligned}$$

Example 12: What are the other three intersection points?

$(0, 0)$ is one obvious other intersection point. For $r = 1 + \sin\theta$, $r = 0$ if $\theta = 3\pi/2$; for $r^2 = 4\sin\theta$, $r = 0$ if $\theta = 0$ or π. But how to get the other two intersection points? Recall that the point with polar coordinates (r, θ) is the same as the point with polar coordinates $(-r, \theta \pm \pi)$. Rewrite the polar equation $r^2 = 4\sin\theta$ as

$$(-r)^2 = 4\sin(\theta \pm \pi) \Leftrightarrow r^2 = 4(-\sin\theta) = -4\sin\theta,$$

for $-\pi \le \theta \le 0$. Now substitute $r = 1 + \sin\theta$:

$$\begin{aligned} (1+\sin\theta)^2 = -4\sin\theta \quad & \Rightarrow \quad 1 + 2\sin\theta + \sin^2\theta = -4\sin\theta \\ & \Rightarrow \quad \sin^2\theta + 6\sin\theta + 1 = 0 \\ & \Rightarrow \quad \sin\theta = -3 \pm 2\sqrt{2} \Rightarrow \sin\theta = -3 + 2\sqrt{2} \\ & \Rightarrow \quad \theta \simeq -9.879 \text{ or } -170.121 \text{ in degrees.} \end{aligned}$$

Example 12, Concluded

If $\sin\theta = -3 + 2\sqrt{2}$, then

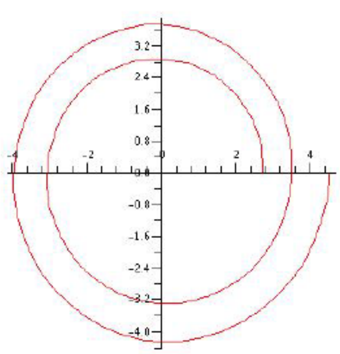
$$\cos\theta = \pm\sqrt{1 - (-3 + 2\sqrt{2})^2} = \pm 2\sqrt{3\sqrt{2} - 4},$$

and $r = 1 + (-3 + 2\sqrt{2}) = -2 + 2\sqrt{2}$. The Cartesian coordinates of the remaining two intersection points are:

$$\begin{aligned} (x, y) &= (r\cos\theta, r\sin\theta) \\ &= \left(\pm 4(\sqrt{2} - 1)\sqrt{3\sqrt{2} - 4}, 14 - 10\sqrt{2}\right) \\ &\simeq (\pm 0.8161427336, -0.14213562) \end{aligned}$$

Example 13: Let $r = e^{\theta/25}$, for $8\pi \le \theta \le 12\pi$.

What is the area within this spiral chamber? For $\theta \in [0, 2\pi]$, let $r_o = e^{(10\pi+\theta)/25}$, $r_i = e^{(8\pi+\theta)/25}$. Then $r_o = a r_i$, for $a = e^{2\pi/25}$, and $r_o^2 - r_i^2 = a^2 r_i^2 - r_i^2 = (a^2 - 1) r_i^2 = (a^2 - 1)\, a^8\, e^{2\theta/25}$. So



$$\begin{aligned} A &= \frac{a^2 - 1}{2} \int_0^{2\pi} a^8\, e^{2\theta/25}\, d\theta \\ &= \frac{(a^2 - 1)\, a^8}{2} \frac{25}{2} \left[e^{2\theta/25} \right]_0^{2\pi} \\ &= \frac{(a^2 - 1)\, a^8}{2} \frac{25}{2} \left(e^{4\pi/25} - 1 \right) \\ &= \frac{(a^2 - 1)\, a^8}{2} \frac{25}{2} (a^2 - 1) \simeq 19.9 \end{aligned}$$