11.10 Taylor Series

We have seen that some functions can be represented as series, which may give valuable information about the function. So far, we have seen only those examples that result from manipulation of our one fundamental example, the geometric series. We would like to start with a given function and produce a series to represent it, if possible.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval of convergence. Then we know that we can compute derivatives of f by taking derivatives of the terms of the series. Let's look at the first few in general:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots$$

By examining these it's not hard to discern the general pattern. The kth derivative must be

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \\ &= k(k-1)(k-2)\cdots(2)(1)a_k + (k+1)(k)\cdots(2)a_{k+1}x + \\ &\qquad + (k+2)(k+1)\cdots(3)a_{k+2}x^2 + \cdots \end{aligned}$$

We can shrink this quite a bit by using factorial notation:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k! a_k + (k+1)! a_{k+1} x + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots$$

Now substitute $x = 0$:

$$f^{(k)}(0) = k! a_k + \sum_{n=k+1}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k! a_k,$$

and solve for a_k:

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Note the special case, obtained from the series for f itself, that gives $f(0) = a_0$.

So if a function f can be represented by a series, we know just what series it is. Given a function f, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called the **Maclaurin series** for f.

EXAMPLE 11.10.1 Find the Maclaurin series for $f(x) = 1/(1-x)$. We need to compute the derivatives of f (and hope to spot a pattern).

$$\begin{aligned} f(x) &= (1-x)^{-1} \\ f'(x) &= (1-x)^{-2} \\ f''(x) &= 2(1-x)^{-3} \\ f'''(x) &= 6(1-x)^{-4} \\ f^{(4)}(x) &= 4!(1-x)^{-5} \\ &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-n-1} \end{aligned}$$

So

$$\frac{f^{(n)}(0)}{n!} = \frac{n!(1-0)^{-n-1}}{n!} = 1$$

and the Maclaurin series is

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n,$$

the geometric series. □

A warning is in order here. Given a function f we may be able to compute the Maclaurin series, but that does not mean we have found a series representation for f. We still need to know where the series converges, and if, where it converges, it converges to $f(x)$. While for most commonly encountered functions the Maclaurin series does indeed converge to f on some interval, this is not true of all functions, so care is required.

As a practical matter, if we are interested in using a series to approximate a function, we will need some finite number of terms of the series. Even for functions with messy derivatives we can compute these using computer software like Sage. If we want to know the whole series, that is, a typical term in the series, we need a function whose derivatives fall into a pattern that we can discern. A few of the most important functions are fortunately very easy.

EXAMPLE 11.10.2 Find the Maclaurin series for $\sin x$.

The derivatives are quite easy: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and then the pattern repeats. We want to know the derivatives at zero: 1, 0, -1, 0, 1, 0, -1, 0,..., and so the Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We should always determine the radius of convergence:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0,$$

so the series converges for every x. Since it turns out that this series does indeed converge to $\sin x$ everywhere, we have a series representation for $\sin x$ for every x. □

Sometimes the formula for the nth derivative of a function f is difficult to discover, but a combination of a known Maclaurin series and some algebraic manipulation leads easily to the Maclaurin series for f.

EXAMPLE 11.10.3 Find the Maclaurin series for $x \sin(-x)$.

To get from $\sin x$ to $x \sin(-x)$ we substitute $-x$ for x and then multiply by x. We can do the same thing to the series for $\sin x$:

$$x \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = x \sum_{n=0}^{\infty} (-1)^n (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+2}}{(2n+1)!}.$$

□

As we have seen, a general power series can be centered at a point other than zero, and the method that produces the Maclaurin series can also produce such series.

EXAMPLE 11.10.4 Find a series centered at -2 for $1/(1-x)$.

If the series is $\sum_{n=0}^{\infty} a_n (x+2)^n$ then looking at the kth derivative:

$$k!(1-x)^{-k-1} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x+2)^{n-k}$$

and substituting $x = -2$ we get $k!3^{-k-1} = k!a_k$ and $a_k = 3^{-k-1} = 1/3^{k+1}$, so the series is

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}.$$

We've already seen this, on page 280. □

Such a series is called the **Taylor series** for the function, and the general term has the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A Maclaurin series is simply a Taylor series with $a = 0$.

Exercises 11.10.

For each function, find the Maclaurin series or Taylor series centered at a, and the radius of convergence.

1. $\cos x \Rightarrow$
2. $e^x \Rightarrow$
3. $1/x$, $a = 5 \Rightarrow$
4. $\ln x$, $a = 1 \Rightarrow$
5. $\ln x$, $a = 2 \Rightarrow$
6. $1/x^2$, $a = 1 \Rightarrow$
7. $1/\sqrt{1-x} \Rightarrow$
8. Find the first four terms of the Maclaurin series for $\tan x$ (up to and including the x^3 term). $\Rightarrow$
9. Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for $x\cos(x^2)$. $\Rightarrow$
10. Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for xe^{-x}. $\Rightarrow$

11.11 Taylor's Theorem

One of the most important uses of infinite series is the potential for using an initial portion of the series for f to approximate f. We have seen, for example, that when we add up the first n terms of an alternating series with decreasing terms that the difference between this and the true value is at most the size of the next term. A similar result is true of many Taylor series.

THEOREM 11.11.1 Suppose that f is defined on some open interval I around a and suppose $f^{(N+1)}(x)$ exists on this interval. Then for each $x \neq a$ in I there is a value z between x and a so that

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

Proof. The proof requires some cleverness to set up, but then the details are quite elementary. We want to define a function $F(t)$. Start with the equation

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + B(x-t)^{N+1}.$$

Here we have replaced a by t in the first $N+1$ terms of the Taylor series, and added a carefully chosen term on the end, with B to be determined. Note that we are temporarily keeping x fixed, so the only variable in this equation is t, and we will be interested only in t between a and x. Now substitute $t = a$:

$$F(a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Set this equal to $f(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Since $x \neq a$, we can solve this for B, which is a "constant"—it depends on x and a but those are temporarily fixed. Now we have defined a function $F(t)$ with the property that $F(a) = f(x)$. Consider also $F(x)$: all terms with a positive power of $(x-t)$ become zero when we substitute x for t, so we are left with $F(x) = f^{(0)}(x)/0! = f(x)$. So $F(t)$ is a function with the same value on the endpoints of the interval $[a, x]$. By Rolle's theorem (6.5.1), we know that there is a value $z \in (a, x)$ such that $F'(z) = 0$. Let's look at $F'(t)$. Each term in $F(t)$, except the first term and the extra term involving B, is a product, so to take the derivative we use the product rule on each of these terms. It will help to write out the first few terms of the definition:

$$\begin{aligned} F(t) = f(t) + \frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \frac{f^{(3)}(t)}{3!}(x-t)^3 + \cdots \\ + \frac{f^{(N)}(t)}{N!}(x-t)^N + B(x-t)^{N+1}. \end{aligned}$$

Now take the derivative:

$$\begin{aligned}
F'(t) = f'(t) &+ \left(\frac{f^{(1)}(t)}{1!}(x-t)^0(-1) + \frac{f^{(2)}(t)}{1!}(x-t)^1\right) \\
&+ \left(\frac{f^{(2)}(t)}{1!}(x-t)^1(-1) + \frac{f^{(3)}(t)}{2!}(x-t)^2\right) \\
&+ \left(\frac{f^{(3)}(t)}{2!}(x-t)^2(-1) + \frac{f^{(4)}(t)}{3!}(x-t)^3\right) + \ldots + \\
&+ \left(\frac{f^{(N)}(t)}{(N-1)!}(x-t)^{N-1}(-1) + \frac{f^{(N+1)}(t)}{N!}(x-t)^N\right) \\
&+ B(N+1)(x-t)^N(-1).
\end{aligned}$$

Now most of the terms in this expression cancel out, leaving just

$$F'(t) = \frac{f^{(N+1)}(t)}{N!}(x-t)^N + B(N+1)(x-t)^N(-1).$$

At some z, $F'(z) = 0$ so

$$\begin{aligned}
0 &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N + B(N+1)(x-z)^N(-1) \\
B(N+1)(x-z)^N &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N \\
B &= \frac{f^{(N+1)}(z)}{(N+1)!}.
\end{aligned}$$

Now we can write

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-t)^{N+1}.$$

Recalling that $F(a) = f(x)$ we get

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1},$$

which is what we wanted to show. ∎

It may not be immediately obvious that this is particularly useful; let's look at some examples.

EXAMPLE 11.11.2 Find a polynomial approximation for $\sin x$ accurate to ± 0.005.

From Taylor's theorem:

$$\sin x = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

What can we say about the size of the term

$$\frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}?$$

Every derivative of $\sin x$ is $\pm\sin x$ or $\pm\cos x$, so $|f^{(N+1)}(z)| \le 1$. The factor $(x-a)^{N+1}$ is a bit more difficult, since $x-a$ could be quite large. Let's pick $a = 0$ and $|x| \le \pi/2$; if we can compute $\sin x$ for $x \in [-\pi/2, \pi/2]$, we can of course compute $\sin x$ for all x.

We need to pick N so that

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < 0.005.$$

Since we have limited x to $[-\pi/2, \pi/2]$,

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < \frac{2^{N+1}}{(N+1)!}.$$

The quantity on the right decreases with increasing N, so all we need to do is find an N so that

$$\frac{2^{N+1}}{(N+1)!} < 0.005.$$

A little trial and error shows that $N = 8$ works, and in fact $2^9/9! < 0.0015$, so

$$\begin{aligned}\sin x &= \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!}x^n \pm 0.0015 \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \pm 0.0015.\end{aligned}$$

Figure 11.11.1 shows the graphs of $\sin x$ and and the approximation on $[0, 3\pi/2]$. As x gets larger, the approximation heads to negative infinity very quickly, since it is essentially acting like $-x^7$. □

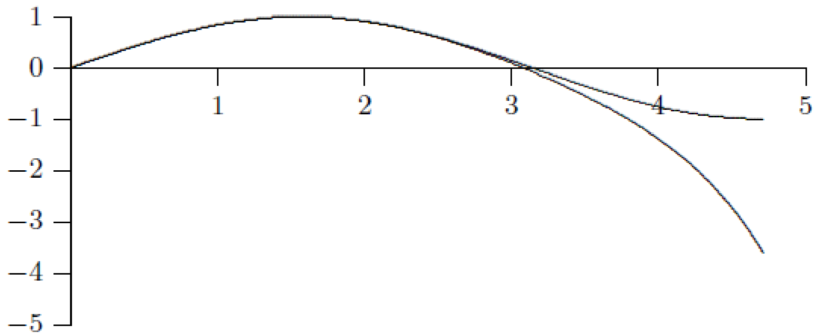

Figure 11.11.1 $\sin x$ and a polynomial approximation. (AP)

We can extract a bit more information from this example. If we do not limit the value of x, we still have

$$\left|\frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}\right| \le \left|\frac{x^{N+1}}{(N+1)!}\right|$$

so that $\sin x$ is represented by

$$\sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n \pm \left|\frac{x^{N+1}}{(N+1)!}\right|.$$

If we can show that

$$\lim_{N\to\infty} \left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

for each x then

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

that is, the sine function is actually equal to its Maclaurin series for all x. How can we prove that the limit is zero? Suppose that N is larger than $|x|$, and let M be the largest integer less than $|x|$ (if $M = 0$ the following is even easier). Then

$$\begin{aligned}\frac{|x^{N+1}|}{(N+1)!} &= \frac{|x|}{N+1}\frac{|x|}{N}\frac{|x|}{N-1}\cdots\frac{|x|}{M+1}\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1}\\ &\le \frac{|x|}{N+1}\cdot 1\cdot 1\cdots 1\cdot\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1}\\ &= \frac{|x|}{N+1}\frac{|x|^M}{M!}.\end{aligned}$$

The quantity $|x|^M/M!$ is a constant, so

$$\lim_{N\to\infty}\frac{|x|}{N+1}\frac{|x|^M}{M!} = 0$$

and by the Squeeze Theorem (11.1.3)

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

as desired. Essentially the same argument works for $\cos x$ and e^x; unfortunately, it is more difficult to show that most functions are equal to their Maclaurin series.

EXAMPLE 11.11.3 Find a polynomial approximation for e^x near $x = 2$ accurate to ± 0.005.

From Taylor's theorem:

$$e^x = \sum_{n=0}^{N} \frac{e^2}{n!}(x-2)^n + \frac{e^z}{(N+1)!}(x-2)^{N+1},$$

since $f^{(n)}(x) = e^x$ for all n. We are interested in x near 2, and we need to keep $|(x-2)^{N+1}|$ in check, so we may as well specify that $|x-2| \le 1$, so $x \in [1,3]$. Also

$$\left|\frac{e^z}{(N+1)!}\right| \le \frac{e^3}{(N+1)!},$$

so we need to find an N that makes $e^3/(N+1)! \le 0.005$. This time $N = 5$ makes $e^3/(N+1)! < 0.0015$, so the approximating polynomial is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{e^2}{120}(x-2)^5 \pm 0.0015.$$

This presents an additional problem for approximation, since we also need to approximate e^2, and any approximation we use will increase the error, but we will not pursue this complication. □

Note well that in these examples we found polynomials of a certain accuracy only on a small interval, even though the series for $\sin x$ and e^x converge for all x; this is typical. To get the same accuracy on a larger interval would require more terms.

Exercises 11.11.

1. Find a polynomial approximation for $\cos x$ on $[0, \pi]$, accurate to $\pm 10^{-3}$ ⇒
2. How many terms of the series for $\ln x$ centered at 1 are required so that the guaranteed error on $[1/2, 3/2]$ is at most 10^{-3}? What if the interval is instead $[1, 3/2]$? ⇒
3. Find the first three nonzero terms in the Taylor series for $\tan x$ on $[-\pi/4, \pi/4]$, and compute the guaranteed error term as given by Taylor's theorem. (You may want to use Sage or a similar aid.) ⇒

4. Show that $\cos x$ is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.
5. Show that e^x is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.

11.12 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

Determine whether the series converges.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+4} \Rightarrow$
2. $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\cdots \Rightarrow$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n^2+4)^2} \Rightarrow$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{8^n} \Rightarrow$
5. $\displaystyle 1-\frac{3}{4}+\frac{5}{8}-\frac{7}{12}+\frac{9}{16}+\cdots \Rightarrow$
6. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+4}} \Rightarrow$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^3(n)}{n^2} \Rightarrow$
8. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^n} \Rightarrow$
9. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} \Rightarrow$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \Rightarrow$
11. $\displaystyle\frac{1}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\frac{3}{4\cdot 5\cdot 6}+\frac{4}{5\cdot 6\cdot 7}+\cdots \Rightarrow$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n)!} \Rightarrow$
13. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{n!} \Rightarrow$
14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \Rightarrow$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^{n-1}}{n!} \Rightarrow$

16. $1 + \frac{5^2}{2^2} + \frac{5^4}{(2 \cdot 4)^2} + \frac{5^6}{(2 \cdot 4 \cdot 6)^2} + \frac{5^8}{(2 \cdot 4 \cdot 6 \cdot 8)^2} + \cdots \Rightarrow$

17. $\displaystyle\sum_{n=1}^{\infty} \sin(1/n) \Rightarrow$

Find the interval and radius of convergence; you need not check the endpoints of the intervals.

18. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \Rightarrow$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+3^n} \Rightarrow$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n} \Rightarrow$

21. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \Rightarrow$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2} x^n \Rightarrow$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} x^{2n} \Rightarrow$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \Rightarrow$

Find a series for each function, using the formula for Maclaurin series and algebraic manipulation as appropriate.

25. $2^x \Rightarrow$

26. $\ln(1+x) \Rightarrow$

27. $\ln\left(\frac{1+x}{1-x}\right) \Rightarrow$

28. $\sqrt{1+x} \Rightarrow$

29. $\frac{1}{1+x^2} \Rightarrow$

30. $\arctan(x) \Rightarrow$

31. Use the answer to the previous problem to discover a series for a well-known mathematical constant. $\Rightarrow$