1 Maclaurin and Taylor Polynomials

1.1 Polynomials

We continue our exploration of sequences and series by introducing something known as a **Taylor Polynomial**.

Taylor Polynomial

A **Taylor Polynomial** centered at a is constructed as follows:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

- To obtain all the coefficients and values, we would take a function, say for example $f(x) = \cos(2x)$ and calculate all the values $f(x), f'(x)$, etc for a center x.
- The center is usually given. A polynomial centered at 0 is also called a Maclaurin polynomial, and we would simply calculate $f(0), f'(0)...$ as necessary.

As practice, let's evaluate the third degree Taylor polynomial (centered at 0) for $f(x) = \cos(x)$.

$f(0) = 1$	$f(x) = \cos(x)$
$f'(0) = 0$	$f'(x) = -\sin(x)$
$f''(0) = -1$	$f''(x) = -\cos(x)$
$f'''(0) = 0$	$f'''(x) = \sin(x)$

Plugging these values in yields: $\boxed{1 - \frac{1}{2}(x^2)}$.

The usefulness of Taylor polynomials is that it allows us to approximate functions like sine and cosine as a polynomial; as we know, derivatives of polynomials are arguable much easier and more straightforward than derivatives of trigonometric functions.

1.2 Inequality

Taylor's Inequality

Taylor's Inequality states:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2}$$

- M is the maximum value of $|f^{(n+1)}(c)|$ for some c in the domain.

2 Power Series

Power Series

A **power series** centered at a is:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ... \tag{3}$$

2.1 Radius of Convergence

Radius of Convergence

For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there exists a value R such that the series converges for $|x-a| < R$. R is called the radius of convergence.

- Let's look at an example. Let's find the radius of convergence for $\sum_{n=0}^{\infty} \frac{nx^n}{10^n}$. To find the radius, we need to use the ratio test. Taking the limit as n goes to infinity, we get L equal to

$$\frac{|x|}{10}$$

- Since we want to know when this converges, we set L = $\frac{|x|}{10}$ less than 1 and solve for x.
- We get $|x| < 10$, so the radius of convergence is 10.

2.2 Interval of Convergence

The **interval of convergence** is very similar to the radius of convergence. Looking back at our previous example, we would check the endpoints to find the interval of convergence. We would test if the series converges at $x = -10$ and $x = 10$. In this case, the series diverges at both points so the interval of convergence is $-10 < x < 10$.

3 Maclaurin and Taylor Series

Taylor Series

A **Taylor Series** is:

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4}$$

where the series is centered at a. A **Maclaurin Series** is a Taylor series centered at 0.

Appendices

A Credits

A.1 Contributions

- Drafted by **Austin Wang**

A.2 Image Credits

- Clock: Veronica Cruz

B Extra Resources/Further Readings

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