RECALL: The k^{th} **Taylor Polynomial** P_k centered at ***c*** for a function *f* is given by …….

$$P(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Definitions of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot \cdot \cdot$$

is called the **Taylor series for $f(x)$ at c.** Moreover, if $c = 0$, then the series is the **Maclaurin series for f.**

Sample Problem #1:

Use the function $f(x) = \frac{1}{x}$ to form the **Taylor Series centered at *c* =*1***. Determine the Radius of Convergence and the Interval of Convergence. **SHOW WORK IN YOUR NOTEBOOK!**

$f(x) = x^{-1}$ $f(1) = 1$

$f'(x) = -x^{-2}$ $f'(1) = -1$

$f''(x) = 2x^{-3}$ $f''(1) = 2$

$f'''(x) = -6x^{-4}$ $f'''(1) = -6$

$f^{IV}(x) = 24x^{-5}$ $f^{IV}(1) = 24$

$f^{(n)}(1) = (-1)^n n!$

$$\boxed{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} |x-1| = |x-1| < 1$$

$\boxed{R = 1}$ $(0, 2)$

at $x=0$: Not included in the original domain of $f(x)$ ∴ $x=0$ is excluded

at $x=2$: $\sum_{n=0}^{\infty} (-1)^n (1)^n = \sum_{n=0}^{\infty} (-1)^n$ Diverge.

$\boxed{\text{Interval of Convergence: } (0,2)}$

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \Sigma\, a_n(x - c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdot \cdot \cdot .$$

Sample Problem #2:

Find a formula for the nth coefficient $\dfrac{f^{(n)}(3)}{n!}$ of the Taylor Series centered at 3 for $f(x)=\dfrac{1}{2x-5}$.

$$f(x) = (2x-5)^{-1}$$

$$f'(x) = -(2x-5)^{-2} \cdot 2$$

$$f''(x) = 2(2x-5)^{-3} \cdot 4$$

$$f'''(x) = -6(2x-5)^{-4} \cdot 8$$

$$f^{iv}(x) = 24(2x-5)^{-5} \cdot 16$$

$$f^{(n)}(x) = (-1)^n n! (2x-5)^{-(n+1)} \cdot 2^n$$

$$\boxed{\text{Coefficient: } (-1)^n 2^n}$$

RECALL: The REMAINDER of a Taylor Approximation of degree n is given by:

$$\boxed{R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}}$$

Where *z* is a value between *x* and *c*.

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

Sample Problem #3:
Set up the Maclaurin series for $f(x)=\sin(x)$. Determine the radius of convergence and interval of convergence.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{iv}(x) = \sin x \qquad f^{iv}(0) = 0$

$$0 + \frac{x}{1!} + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!} + \frac{0x^6}{6!} - \frac{x^7}{7!} \cdots = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}}$$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+2} x^{2n+1}}{(2n+1)!}}{\dfrac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)\, x^2}{(2n+1)(2n)} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n)} = 0 < 1$$

$\boxed{R = \infty} \Rightarrow$ Interval of Convergence: $(-\infty, \infty)$

$$R_n \le \frac{1}{(2n+1)!} x^{2n+1}$$

$$\lim_{n \to \infty} R_n \le \lim_{n \to \infty} \frac{x^{2n+1}}{(2n+1)!} = 0 \Rightarrow$$

The series converges to $f(x) = \sin x$ for $\mathbb{R}$.

Guidelines for Finding a Taylor Series

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), f'(c), f''(c), f'''(c), \cdot \cdot \cdot, f^{(n)}(c), \cdot \cdot \cdot$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdot \cdot \cdot .$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Sample Problem #4:

Set up the Maclaurin series for $f(x) = \sin\left(x^2\right)$.

SHOW WORK IN YOUR NOTEBOOK!

From Sample Problem #3: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}$

therefore $\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} (x^2)^{2n-1}$

$$\boxed{\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{4n-2}}$$

Sample Problem #5:

Set up the Maclaurin series for $f(x) = e^{5x}$. Determine for which values of x this series equals $f(x) = e^{5x}$.

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$$\left.\begin{array}{ll} g(x)=e^x & g(0)=1 \\ g^{(n)}(x)=e^x & g^{(n)}(0)=1 \end{array}\right\} \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$f(x)=e^{5x} \longrightarrow \sum_{n=0}^{\infty} \frac{1}{n!}(5x)^n = \boxed{\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{5^{n+1}x^{n+1}}{(n+1)!}}{\frac{5^n x^n}{n!}} \right| = \lim_{n\to\infty} \frac{5x}{n+1} = 0 < 1 \Rightarrow \boxed{R=\infty}$$

$$\boxed{e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n \text{ for all } \mathbb{R}.}$$

Sample Problem #6:

Set the Taylor series centered at $\frac{\pi}{2}$ for $f(x) = \cos(x)$. Then determine for which *x* values this series equals $f(x) = \cos(x)$.

SHOW WORK IN YOUR NOTEBOOK!

HINT: Odd numbers have the form 2n+1 and evens have the form 2n. So write the formula for $f^{(2n)}\left(\frac{\pi}{2}\right)$ *and another formula for* $f^{(2n+2)}\left(\frac{\pi}{2}\right)$.

$$f(x) = \cos x \qquad f(\pi/2) = 0$$
$$f'(x) = -\sin x \qquad f'(\pi/2) = -1$$
$$f''(x) = -\cos x \qquad f''(\pi/2) = 0$$
$$f'''(x) = \sin x \qquad f'''(\pi/2) = 1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi/2)^{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-\pi/2)^{2n+3}}{(2n+3)!}}{\frac{(-1)^{n+1}(x-\pi/2)^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)\,x^2}{(2n+3)(2n+2)}\right| = \underbrace{0<1}_{\boxed{R=\infty}}$$

Interval of Convergence: $(-\infty, \infty)$

$$R_n \le \frac{1}{(n+1)!}(x-\pi/2)^{n+1} \qquad \underbrace{\lim_{n\to\infty} R_n \le \lim_{n\to\infty}\frac{(x-\pi/2)^{n+1}}{(n+1)!} = 0}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2n+1)!}(x-\pi/2)^{2n+1} \text{ for } \mathbb{R}.}$$