9-3 Taylor's Theorem & Lagrange Error Bounds

Actual Error

This is the real amount of error, not the error bound (worst case scenario). It is the difference between the actual f(x) and the polynomial.

Steps:

1. Plug x-value into f(x) to get a value. f(a)
2. Plug x-value into the polynomial and get another value. P(a)
3. The absolute difference between the two is the error. |f(a) – P(a)| = Actual Error

Example 1:

Given $f(x) = \frac{1}{1-x}$, approximate f(0.1) using a 2nd degree Taylor polynomial and find the error.

For example: If you are trying to find the error of a 2nd degree Taylor polynomial approximation of f(x) = $\frac{1}{1-x}$, you must first find the 3rd derivative, because the formula uses $f^{n+1}(z)$, not $f^{n}(z)$.

f'(x) = $\frac{1}{(1-x)^2}$, f''(x) = $\frac{2}{(1-x)^3}$, and f'''(x) = $\frac{6}{(1-x)^4}$

Also, for this function, x = 0.1 and a = 0. Plug these two values into the 3rd derivative.

$$f^{3}(0) = \frac{6}{(1-0)^4} = 6$$

$f^{3}(.1) = \frac{6}{(1-.1)^4} =$ **9.145** ← **this is bigger!**

Next, plug in 9.145 for $f^{n+1}(z)$ in the La Grange formula:

Error bound = $\frac{9.145(.1-0)^3}{3!}$ = **.00152**

Exception!

⇓

When f(x) = sin(x) or cos(x), the value for $f^{n+1}(z)$ will always be equal to 1, because that is the greatest value of any sin or cos function.

Example 2:

What is the error for the fourth degree polynomial approximation of cos x when x = $\frac{\pi}{4}$

Example 3

Find a formula for the truncation error if we use $1+x^2+x^4+x^6$ to approximate $\frac{1}{1-x^2}$ on (-1,1).

Taylor's Formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

If $R_n(x) \to 0$ as $n \to \infty$, then the Taylor series converges to f(x) on the interval I for all x on I.

Lagrange Formula

This method uses a special form of the Taylor formula to find the **error bound** of a polynomial approximation of a Taylor series.

$$R_n(x) = \frac{f^{(n+1)}(z)(x-a)^{(n+1)}}{(n+1)!} = \text{error bound}$$

Where:

- a is where the series is centered
- z is a value between a and x (z is usually a or x)

The variable z is a number between x and a (z giving the largest value for $f^{(n+1)}(z)$), but to find the error bound, z ends up being equal to either a or x. To determine whether the z value will be the same as x or a, you must plug each number into $f^{n+1}(z)$ to see which gives the greatest number.

For sin or cos $f^{(n+1)}(z) = 1$, (even if all z's give smaller values).

Polynomial value $\pm$ error bound = range of possible values of the series.

Examples for La Grange error bound:

a.) Find the upper bound for the error for the 5th degree polynomial approximation of *e*.

e is equal to e^1, whose series can be determined from the Taylor series of e^x.

$$e^1 = \frac{(1)^0}{0!} + \frac{(1)^1}{1!} + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!} + \frac{(1)^5}{5!}$$

The La Grange formula is,

$$\frac{f^{(6)}(z)(x)^6}{6!}$$

All derivatives of e^1 are e^1, so $f^6(z) = e^z$

To find *z*, plug in the values for *a* and *x* into e^z.

$a = 0$, $e^0 = 1$
$x = 1$, $e^1 = e$

$e > 1$, so $z = 1$

$$\frac{e^1(1)^6}{6!} = \frac{e}{6!} = \frac{e}{720} = .00377$$

The actual error for this 5th degree polynomial falls somewhere between the real value of e^1 and $e^1 + .00377$.

$e^1 = 2.71828$

$$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.71666$$

The error is 2.71828 – 2.71666, which equals 0.00162. This number is less than the upper bound for the error, 0.00377, which shows how the La Grange formula works.

b.) What degree Taylor polynomial for ln(1.2) might have an error less than 0.001? (In other words, the upper bound for the error would be 0.001)

First, start off with the La Grange formula, whose value must be less than 0.001:

$$\frac{f^{n+1}(z)(x-a)^{n+1}}{(n+1)!} < 0.001$$

For the function ln(x), a =1

and in this case, x =1.2

The derivatives of ln(x) are as follows:

$$f(x)=\ln(x), f'(x)=\frac{1}{x}, f''(x)=\frac{-1}{x^2}, f'''(x)=\frac{2}{x^3}, f''''(x)=\frac{-6}{x^4}$$

Since you don't know the value of *n*, a general formula for the $n+1$ derivative must be used. The formula for the *nth* derivative can be obtained from above and is as follows:

$$f^n(x)=\frac{(-1)^{n+1}(n-1)!}{x^n}$$

To find $f^{n+1}(x)$, simply substitute $n+1$ for *n* into that equation:

$$f^{n+1}(x)=\frac{(-1)^{n+2}(n)!}{x^{n+1}}$$

This is what you will put into the La Grange formula for $f^{n+1}(z)$, changing *x* to *z*.

Still, you must find the value for z. It will be equal to either *a* or *x*. When plugging the two values into the above list of derivative for ln(x), you find that 1 always produces the greater value, so z = 1.

Now, the error bound formula looks something like this:

$$\frac{\frac{(-1)^{n+2}(n)!}{z^{n+1}}(x-a)^{n+1}}{(n+1)!}=\frac{\frac{(n)!}{z^{n+1}}(1.2-1)^{n+1}}{(n+1)!}=\frac{n!}{(n+1)!}\frac{(.2)^{n+1}}{z^{n+1}}=\frac{1}{n+1}\left(\frac{.2}{z}\right)^{n+1}<0.001$$

$$\frac{1}{n+1}(.2)^{n+1}<0.001$$

Next you must simply use the concept of trial and error. Choose values for *n* and keep plugging them in to the inequality. When the term on the left ends up being greater than 0.001, you know that you have crossed the line and your value for *n* will be the previous number (before the value exceeded 0.001).

$$\frac{1}{3+1}(.2)^{3+1}=.0004<.001$$

$$\frac{1}{2+1}(.2)^{2+1}=.00267>.001$$

The value for *n* is 3.

Ex 1:
Find the 3rd degree polynomial approximation for e^x at 1, centered at 0.
Find the range of possible values for e^x at 1, centered at 0. Use the Lagrange error equation.

Ex 2:
Find the 4th degree Maclaurin polynomial approximation for cos(x) where a = 0, evaluated at 1.
Find the Lagrange error bound.

Ex 3:
Use graphs to find a Taylor Polynomial $P_n(x)$ for cos x so that $| P_n(x) - \cos(x)| < 0.001$ for every x in $[-\Pi,\Pi]$.