$$R_n(x) \le \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right| \quad \text{for some } z \text{ between } x \text{ and } c.$$

Examples:

1. Find the Lagrange error bound in approximating sin(0.1) using the Maclaurin polynomial of degree 3.

$c=0$ $x=.1$ $[0,.1]$

$f'(x)=\cos x$

$f''(x)=-\sin x$

$f'''(x)=-\cos x$

$f^{(4)}(x)=\sin x$

$\left| \frac{\sin z}{4!}(.1)^4 \right|$

$= \frac{\sin .1}{4!}(.1)^4 = 4.1597\times10^{-7}$

2. Find the Lagrange error bound in approximating $e^{-0.5}$ using the Maclaurin polynomial of degree 4.

$c=0$ $x=-.05$ $[-.05,0]$

$f(x)=e^x$

$f^{(5)}(x)=e^4$

$\left| \frac{e^z}{5!}(-.5)^5 \right|$

$\left| \frac{e^0}{5!}(-.5)^5 \right| = 2.604\times10^{-4}$

3. How many terms of the Maclaurin polynomial of $f(x)=\ln(1+x)$ are needed to approximate f(-0.2) with an error less than 10^{-3}.

$c=0$ $x=-.2$

$f'(x)=\frac{1}{1+x}$

$f''(x)=\frac{-1}{(1+x)^2}$

$f'''(x)=\frac{2}{(1+x)^3}$

$f^{(4)}(x)=\frac{-6}{(1+x)^4}$

$f^{n}(x)=\frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$

$\left| \frac{\frac{(n-1)!}{(1+x)^n}}{(n+1)!}(-.2)^{n+1} \right| < .001$

$= \left| \frac{(-.2)^{n+1}}{(n+1)n(.8)^n} \right| < .001$

$n=3$ $.0002$

$n=2$ $.0021$

so $n=3$

4. How many terms of the Maclaurin polynomial of $f(x)=\cos x$ centered are required to approximate f(0.7) with an error less than 10^{-4}.

$\left| \frac{1}{(n+1)!}(.7)^{n+1} \right| < .001$

$n=5$ $.00016$

$n=6$ $.00001634$

$n=6$

Homework: $\approx 2.71\overline{6}$

1. Use Taylor's Theorem to obtain an upper bound for the error of the approximation for $e \approx 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\frac{1^4}{4!}+\frac{1^5}{5!}$. Then calculate the exact value of the error. e^x $x=1$ $[0,1]$

$\left|\frac{e^1}{6!}(1)^6\right| = .0038$ upperbound

actual $e-2.71\overline{6} = .001615$

2. Determine the degree of the Maclaurin polynomial required for the error in the approximation of the f(x)=sinx at x =0.3 to be less than 0.001. $[0,.3]$

$\left|\frac{1}{(n+1)!}(.3)^{n+1}\right| < .001$

$n=2$ $.0045$ $n=3$

$n=3$ $.0003375$

3. Determine the degree of the Maclaurin polynomial required for the error in the approximation of the f(x)=ln(x+1) at x = 0.5 to be less than 0.0001. $[0,.5]$

$\left|\frac{\frac{(n-1)!}{1.5^n}}{(n+1)!}(.5)^{n+1}\right| < .0001$ $\left|\frac{.5^{n+1}}{(n+1)n(1.5)^n}\right|$

$n=3$ $.0015$

$n=4$ $.003$

$n=5$ $n=5$ $.000068$

4. Find the Lagrange error bound in approximating cos(0.45) using the Maclaurin polynomial of degree 3.

$[0,.45]$

$f'(x)=-\sin x$ $\left|\frac{\cos 0}{4!}(.45)^4\right| = .0017086$

$f''(x)=-\cos x$

$f'''(x)=\sin x$

$f^{(4)}(x)=\cos x$

5. Find the Lagrange error bound in approximating $f(x)=\ln(1+x)$ at $x=0.6$ Maclaurin polynomial of degree 3. $[0,.6]$

$f^{(4)}(x)=\frac{-6}{(1+x)^4}$ $\left|\frac{\frac{-6}{(1.6)^4}}{4!}(.6)^4\right| = \left|\frac{.375^4}{4}\right| = .00494$