Lagrange Remainder Worksheet

Use the Remainder Theorem to bound the error involved in using the specific Taylor polynomial, centered at 0, to approximate *f(x)* at the given value.

1. $P_5(x)$ for $f(x) = \cos x$ at $x = 0.2$

$P_5(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_5(0.2) = 0.98006666$

$\left|R_5(0.2)\right| \le \left|\frac{(0.2)^6}{6!} \cdot \max f^{(6)}(z)\right|$

$\le 8.888 \times 10^{-8}$

$0 < z < 0.2$

$\max f^{(6)}(z) = 1$

2. $P_4(x)$ for $f(x) = e^x$ at $x = 0.8$

$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$P_4(0.8) = 2.2224$

$\left|R_4(0.8)\right| \le \left|\frac{(0.8)^5}{5!} \cdot \max f^{(5)}(z)\right|$

≤ 0.0060772104

$0 < z < 0.8$

$\max f^{(5)}(z) = e^{0.8}$

Use the Remainder Theorem to bound the error involved in using the specific Taylor polynomial, at the given center, to approximate *f(x)* at the given value.

3. $P_2(x)$ for $f(x) = x^{5/2}$, centered at 1. Approximate *f(1.7)*.

$P_2(x) = 1 + \frac{5}{2}(x-1) + \frac{15}{8}(x-1)^2$

$P_2(1.7) = 3.66875$

$\left|R_2(1.7)\right| \le \left|\frac{(0.7)^3}{3!} \cdot \max f^{(3)}(z)\right|$

≤ 0.1071875

$1 < z < 1.7$

$\max f^{(3)}(z) = \frac{15}{8}$

4. $P_3(x)$ for $f(x) = \frac{1}{1-x}$, centered at 2. Approximate *f(2.4)*.

$P_3(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3$

$P_3(2.4) = -0.696$

$\left|R_3(2.4)\right| \le \left|\frac{(0.4)^4}{4!} \cdot \max f^{(4)}(z)\right|$

≤ 0.0256

$2 < z < 2.4$

$\max f^{(4)}(z) = 24$

Determine the degree of the Taylor polynomial, centered at 0, that would be required to approximate the function at the given point to within the stated accuracy.

5. $f(x) = x\ln(1+x)$, at $x = -0.2$, within $1/100$

$$f(x) = x\ln(1+x)$$

$$f'(x) = x(1+x)^{-1} + \ln(1+x)$$

$$f''(x) = -x(1+x)^{-2} + (1+x)^{-1} + (1+x)^{-1} = -x(1+x)^{-2} + 2(1+x)^{-1}$$

$$f'''(x) = 2x(1+x)^{-3} - (1+x)^{-2} - 2(1+x)^{-2} = 2x(1+x)^{-3} - 3(1+x)^{-2}$$

$$f^{(4)}(x) = -6x(1+x)^{-4} + 2(1+x)^{-3} + 6(1+x)^{-3} = -6x(1+x)^{-4} + 8(1+x)^{-3}$$

$$f^{(5)}(x) = 24x(1+x)^{-5} - 6(1+x)^{-4} - 24(1+x)^{-4} = 24x(1+x)^{-5} - 30(1+x)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\cdot\frac{(n-1)!\cdot x}{(1+x)^n} + (-1)^n\left[\frac{(n-2)!}{(1+x)^{n-1}} + \frac{(n-1)!}{(1+x)^{n-1}}\right] = (-1)^{n+1}\cdot\frac{x(n-1)!}{(1+x)^n} + (-1)^n\left[(n-2)!\left(\frac{n}{(1+x)^{n-1}}\right)\right]$$

$$= (-1)^n\left[\frac{-x(n-1)!}{(1+x)^n} + \frac{(n-2)!}{(1+x)^{n-1}} + \frac{(n-1)!}{(1+x)^{n-1}}\right] = (-1)^n\left[\frac{-x(n-1)!}{(1+x)^n} + \frac{n(n-2)!}{(1+x)^{n-1}}\right]$$

$$f^{(n+1)}(x) = (-1)^{n+1}\left[\frac{-x\cdot n!}{(1+x)^{n+1}} + \frac{(n-1)!}{(1+x)^n} + \frac{n!}{(1+x)^n}\right] = (-1)^{n+1}\left[\frac{-x\cdot n!}{(1+x)^{n+1}} + \frac{(n+1)(n-1)!}{(1+x)^n}\right]$$

$$|R_n(-0.2)| \le \left|\frac{(-0.2)^{n+1}}{(n+1)!}\cdot\left[\frac{-z\cdot n!}{(1+z)^{n+1}} + \frac{(n+1)(n-1)!}{(1+z)^n}\right]\right|$$

$$\le \left|\frac{1}{5^{n+1}}\left[\frac{-z}{(n+1)(1+z)^{n+1}} + \frac{1}{n(1+z)^n}\right]\right|$$

$$-0.2 < z < 0$$

$f^{(n+1)}(x)$ max @ $x = -0.2$

$$\le \left|\frac{1}{5^{n+1}}\left[\frac{\frac{1}{5}}{(n+1)\left(\frac{4}{5}\right)^{n+1}} + \frac{1}{n\left(\frac{4}{5}\right)^n}\right]\right|$$

$$\le \left|\frac{1}{20(n+1)\cdot 4^n} + \frac{1}{5n\cdot 4^n}\right| < \frac{1}{100} \Rightarrow n \ge 2$$

6. $f(x) = e^{2x}$, at $x = 0.5$, within $1/100$

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$\vdots$

$f^{(n)}(x) = 2^n e^{2x}$

$f^{(n+1)}(x) = 2^{n+1} e^{2x}$

$$\left|R_n(0.5)\right| \le \left| \frac{(0.5)^{n+1}}{(n+1)!} \cdot \left[2^{n+1} e^{2z}\right] \right|$$

$0 < z < 0.5$

max $f^{(n+1)}(x)$ @ $x = 0.5$

$$\le \left| \frac{1}{2^{n+1}(n+1)!} \cdot 2^{n+1} e \right|$$

$$\le \left| \frac{e}{(n+1)!} \right| < \frac{1}{100} \Rightarrow n \ge 5$$

2004 BC6 parts (a) and (c)

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

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(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

<u>2004 BC 6 :</u>

(a) $$P_3(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} x - \frac{25\sqrt{2}}{2} \cdot \frac{x^2}{2!} - \frac{125\sqrt{2}}{2} \cdot \frac{x^3}{3!}$$

(b) $$\left|R_3\left(\tfrac{1}{10}\right)\right| \le \left| \frac{\left(\frac{1}{10}\right)^4}{4!} \cdot \max f^{(4)}(z) \right| \quad 0 < z < \tfrac{1}{10} \quad \max f^{(4)}(z) = 625$$

$$\le \frac{1}{10^4} \cdot \frac{1}{4!} \cdot 625 = \frac{1}{384} < \frac{1}{100}$$

1999 BC4 parts (a) and (b)

The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

1999 BC 4:

(a) $P_3(x) = -3 + 5(x-2) + 3 \cdot \frac{(x-2)^2}{2!} - 8 \cdot \frac{(x-3)^3}{3!}$

$P_3(1.5) = -4.958$

(b) $|R_3(1.5)| \le \left| \frac{(1.5-2)^4}{4!} \cdot \max f^{(4)}(z) \right|$

$\le \frac{(-.5)^4}{4!} \cdot 3 = 0.0078125$

So, $f(1.5) > -4.958 - 0.0078125 > -4.966 > -5$

Therefore, $f(1.5) \ne -5$