AP Calculus BC

Worksheet – Taylor's Theorem and Lagrange Error Bounds

Calculator OK on problems marked with an asterisk (*).

*1. What is the smallest order of Taylor polynomial centered at $x = 1$ which will approximate e^{x-1} on the domain $-1 \le x \le 3$ with LaGrange error bound less than 1?

$f(1) = 1$
$f'(1) = 1$

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

$\left| e^2 \cdot \frac{(2)^{n+1}}{(n+1)!} \right| < 1$

$f(x) = 1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \ldots$

$n = 5 \to R(x) = 0.657$

$e^2 = 7.389$

*2. The hyperbolic sine is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$. A third-order Taylor polynomial approximation is $\sinh x \approx x + \frac{x^3}{3!}$. If this is used to approximate $\sinh x$ for $|x| \le 2$, which is the LaGrange error bound?

$a = 0$
$n = 3$

(A) 4.836 (B) 3.627 (C) 2.718 (D) 2.508 (E) 2.418

$f'(x) = \frac{e^x + e^{-x}}{2}$
$f''(x) = \frac{e^x - e^{-x}}{2}$
$f'''(x) = \frac{e^x + e^{-x}}{2}$

$R \le \frac{e^2 - e^{-2}}{2} \cdot \frac{(2-0)^4}{4!}$

M
$f^{(4)}(x) = \frac{e^x - e^{-x}}{2}$

The Taylor series for $\cos x$ about $x = 0$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$. If h is a function such that $h'(x) = \cos x^3$, then the coefficient of x^7 in the Taylor series for $h(x)$ about $x = 0$ is

(A) $-\frac{1}{14}$ (B) $-\frac{1}{7!}$ (C) 0 (D) $\frac{1}{7!}$ (E) $\frac{1}{14}$

$h'(x) = \cos x^3$

$h(x) = x - \frac{x^7}{14}$

$h'(x) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \ldots$

*4. The Taylor series for e^x, centered at $x = 0$, is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let f be the function given by the sum of the first four nonzero terms of this series. The maximum value of $|e^x - f(x)|$ for $-0.6 \le x \le 0.6$ is

(A) 0.0036 (B) 0.0048 (C) 0.0052 (D) 0.0061 (E) 0.0081

$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$f(-0.6) = 0.544$

$f(0.6) = 1.816$ $e^{0.6} = 1.8221$ $e^{-0.6} = 0.5488$

5. *True or False*. If $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \ldots$ is the Maclaurin series for the function $f(x)$, then $f'(0) = 1$. Justify your answer.

True, the coefficient of the x term is 1, which is $f'(0)$

$f(0)=1$
$f'(0)=0$
$f''(0)=-1$
$f'''(0)=0$

*6. Let $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ be the Maclaurin series for $\cos x$. Which of the following gives the smallest value of n for which $|P_n(x) - \cos x| < 0.01$ for all x in the interval $[-\pi, \pi]$?

Max error at $\cos \pi = -1$

(A) 12 (B) 10 (C) 8 (D) 6 (E) 4

$-1.01 < P_n(x) < -0.99$
$P_{10}(x) = -1.00183$

7. Which of the following is the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 1$?

$f(1)=1$
$f'(1) = -\frac{1}{x^2} = -1$
$f''(1) = \frac{2}{x^3} = 2$
$f'''(1) = -\frac{6}{x^4} = -6$

(A) $\sum_{n=0}^{\infty} (x-1)^n$ (B) $\sum_{n=0}^{\infty} (-1)^n x^n$ (C) $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

(D) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n!}$ (E) $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

$1-(x-1)+\frac{2(x-1)^2}{2!}-\frac{6(x-1)^3}{3!}$

$f'(x) = \frac{1}{1+x}$
$f''(x) = -\frac{1}{(1+x)^2}$

*8. The approximation $\ln(1+x) \approx x - \frac{x^2}{2}$ is used when x is small. Use the Lagrange form of the remainder to get a bound for the maximum error when $|x| \le 0.1$

$f'''(x) = \frac{2}{(1+x)^3} \rightarrow$ Max value is at $x = -0.1$, so $M = \left| \frac{2}{(1+(-0.1))^3} \right| = \frac{2000}{729}$

$|R_2(x)| \le \frac{2000}{729} \cdot \frac{(0.1)^3}{3!} < \boxed{0.00046}$

9. Prove that $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges to $\cos x$ for all real x.

$|f^{(n+1)}(c)| \le 1$
$M = 1$

$a = 0$

$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-0)^{n+1} \right| \le \frac{1}{(n+1)!} x^{n+1}$

$|R_n(x)| \le \frac{x^{n+1}}{(n+1)!}$

$\boxed{\text{R} \to 0 \text{ as } n \to \infty \text{, therefore series converges to } \cos x.}$

*10. Let $T_3(x)$ be the third order Taylor polynomial for $f(x) = \ln x$ at $a = 1$. Find a bound for the error $|T_3(1.2) - \ln(1.2)|$.

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{6}{x^4}$

$f^{(4)}(1) = -6 \quad f^{(4)}(1.2) = -2.894$

$|f^{(4)}(1)| > |f^{(4)}(1.2)|$, so use $|f^{(4)}(1)|$ for M,

so $M = 6$

$|T_3(1.2) - \ln(1.2)| \le M \cdot \frac{|x-1|^4}{4!} = 6\,\frac{(1.2-1)^4}{24} \approx \boxed{0.0004}$

*11. Let $f(x) = \sqrt{1+x}$ and let $T_n(x)$ be the Taylor polynomial centered at $a = 8$.

a. Find $T_3(x)$ and calculate $T_3(8.02)$.

$f(8) = 3$

$f'(8) = \frac{1}{2\sqrt{1+x}} = \frac{1}{6}$

$f''(8) = -\frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4}\left(\frac{1}{27}\right) = -\frac{1}{108}$

$f'''(8) = \frac{3}{8}(1+x)^{-5/2} = \frac{3}{8}\left(\frac{1}{243}\right) = \frac{1}{648}$

$\boxed{T_3(x) = 3 + \frac{x-8}{6} - \frac{(x-8)^2}{216} + \frac{(x-8)^3}{3888}}$

$T_3(8.02) = 3 + \frac{.02}{6} - \frac{(.02)^2}{216} + \frac{(.02)^3}{3888} = \boxed{3.00333}$

b. Find a bound for $|T_3(8.02) - \sqrt{9.02}|$.

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$ ← strictly increasing so max value will be at $x = 8.02$

$f^{(4)}(8.02) = -.00042535$ ← use this for M (absolute value of this)

$|T_3(8.02) - \sqrt{9.02}| \le .00042535\,\frac{(8.02-8)^4}{4!} \approx \boxed{2.836 \times 10^{-12}}$

*12. Which of the following gives the Taylor polynomial of order 5 approximation to $\sin(1.5)$?

(A) 0.965 (B) 0.985 (C) 0.997 (D) 1.001 (E) 1.005

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sin(1.5) = 1.5 - \frac{(1.5)^3}{3!} + \frac{(1.5)^5}{5!}$