CALCULUS BC
WORKSHEET ON POWER SERIES AND LAGRANGE ERROR BOUND

Work the following on **notebook paper**. Use your calculator on problem 1 only.

1. Let f be a function that has derivatives of all orders for all real numbers x Assume that

$$f(5)=6,\ f'(5)=8,\ f''(5)=30,\ f'''(5)=48,\ \text{and}\ \left|f^{(4)}(x)\right| \le 75$$

for all x in the interval $[5, 5.2]$.

(a) Find the third-degree Taylor polynomial about $x=5$ for $f(x)$.

(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.

(c) Find an interval $[a, b]$ such that $a \le f(5.2) \le b$. Give three decimal places.

(d) Could $f(5.2)$ equal 8.254? Show why or why not.

2. Let f be the function given by $f(x)=\cos\left(2x+\frac{\pi}{6}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right| < \frac{1}{12,000}$.

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3. Find the first four nonzero terms of the power series for $f(x)=\sin x$ centered at $x=\frac{3\pi}{4}$.

Find the first four nonzero terms and the general term for the Maclaurin series for each of the following, and find the interval of convergence for each series.

4. $f(x)=x\cos\left(x^3\right)$

5. $g(x)=\dfrac{x^2}{1+x}$

Find the radius and interval of convergence for:

6. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{3^n n^2}$

7. $\displaystyle\sum_{n=1}^{\infty}(2n)!(x-5)^n$

Multiple Choice.

8. The coefficient of x^6 in the Taylor series expansion about $x=0$ for $f(x)=\sin\left(x^2\right)$ is

$(A)\ -\frac{1}{6}$ $\quad (B)\ 0$ $\quad (C)\ \frac{1}{120}$ $\quad (D)\ \frac{1}{6}$ $\quad (E)\ 1$

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9. If f is a function such that $f'(x)=\sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x=0$ is

$(A)\ \frac{1}{7!}$ $\quad (B)\ \frac{1}{7}$ $\quad (C)\ 0$ $\quad (D)\ -\frac{1}{42}$ $\quad (E)\ -\frac{1}{7!}$

Answers

<u>Answers to Worksheet on Power Series and Lagrange Error Bound</u>

1. (a) $6+8(x-5)+15(x-5)^2+8(x-5)^3$

   (b) $f(5.2)\approx P_3(5.2)=8.264$

   $|R_3(5.2)|\le 0.005$

   (c) $8.259\le f(5.2)\le 8.269$

   (d) No, 8.254 does not lie in the interval found in part (c).

2. (a) $\dfrac{\sqrt{3}}{2}-x-\dfrac{2\sqrt{3}}{2!}x^2+\dfrac{4}{3!}x^3$

   (b) $\left|R_3\left(\dfrac{1}{10}\right)\right|\le\left|\dfrac{16\left(\dfrac{1}{10}\right)^4}{4!}\right|=\dfrac{2^4\left(\dfrac{1}{2^4\cdot5^4}\right)}{4!}=\dfrac{1}{5^4\cdot4!}=\dfrac{1}{625\cdot24}=\dfrac{1}{15,000}<\dfrac{1}{12,000}$

3. $\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\left(x-\dfrac{3\pi}{4}\right)-\dfrac{\sqrt{2}}{2\cdot2!}\left(x-\dfrac{3\pi}{4}\right)^2+\dfrac{\sqrt{2}}{2\cdot3!}\left(x-\dfrac{3\pi}{4}\right)^3+\ldots$

4. $x-\dfrac{x^7}{2!}+\dfrac{x^{13}}{4!}-\dfrac{x^{19}}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{6n+1}}{(2n)!}$. Converges for all real numbers.

5. $x^2-x^3+x^4-x^5+\ldots=\sum_{n=0}^{\infty}(-1)^n x^{n+2}$. Converges for $-1<x<1$.

6. Radius = 3; interval: $-1\le x\le 5$

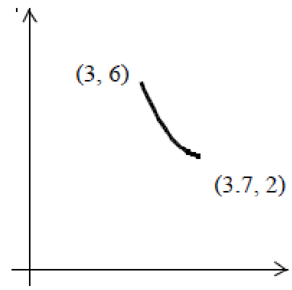
7. Converges only if $x=5$

8. A

9. D

CALCULUS BC
WORKSHEET ON SERIES AND ERROR
Work the following on **notebook paper**. You may use your calculator on problems 1, 2, 3, and 6.

1. Let f be a function that has derivatives of all orders. Assume $f(3)=1$, $f'(3)=\frac{1}{2}$, $f''(3)=-\frac{1}{4}$, $f'''(3)=\frac{3}{8}$, and the graph of $f^{(4)}(x)$ on [3, 3.7] is shown on the right. The graph of $f^{(4)}(x)$ is decreasing on [3, 3.7].
   (a) Find the third-degree Taylor polynomial about $x=3$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(3.7)$.
   (c) What is the maximum possible error for the approximation made in part (b)?
   (d) Could $f(3.7)$ equal 1.283? Show why or why not.



Graph of $f^{(4)}(x)$

2. Let f be the function defined by $f(x)=\sqrt{x}$.
   (a) Find the second-degree Taylor polynomial about $x=4$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(5.1)$.
   (c) Use the Lagrange error bound to find a bound on the error for the approximation in part (b).
   (d) Find the value of $\left|f(5.1)-P_2(5.1)\right|$.

3. Find the maximum error incurred by approximating the sum of the series $1-\frac{1}{2!}+\frac{2}{3!}-\frac{3}{4!}+\frac{4}{5!}-\ldots+(-1)^{n+1}\frac{n-1}{n!}+\ldots$ by the sum of the first five terms. Justify your answer.

4. Let f be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x=0$.
   (a) Find $P(x)$.
   (b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

5. Use series to find an estimate for $\int_0^1 e^{-x^2}\,dx$ so that the error is less than $\frac{1}{200}$. Justify your answer.

6. Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x=1$. If the maximum value of the fifth derivative between $x=1$ and $x=3$ is 0.01, that is, $\left|f^{(5)}(x)\right|<0.01$, find the maximum error incurred using this approximation to compute $f(3)$.

7. The Taylor series about $x=5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=5$ is given by

$$f^{(n)}(5)=\frac{(-1)^n n!}{2^n(n+2)} \text{ and } f(5)=\frac{1}{2}.$$

(a) Write the third-degree Taylor polynomial for f about $x=5$.
(b) Show that the third-degree Taylor polynomial for f about $x=5$ approximates $f(6)$ with an error less than 0.02.

Answers

Answers to Worksheet on Series and Error

1. (a) $1+\frac{x-3}{2}-\frac{(x-3)^2}{4\cdot 2!}+\frac{3(x-3)^2}{8\cdot 3!}$

   (b) 1.310

   (c) Since $f^{(4)}(x)$ is decreasing on [3, 3.7], the maximum value of $f^{(4)}(x)$ is $f^{(4)}(3)=6$

   so $|\text{Error bound}|\le\left|\frac{6(3.7-3)^4}{4!}\right|=0.060.$

   (d) Yes, $1.250\le f(3.7)\le 1.370$ so $f(3.7)$ could equal 1.283.

2. (a) $2+\frac{x-4}{4}-\frac{(x-4)^2}{32\cdot 2!}$

   (b) 2.256

   (c) The maximum value of the third derivative $f'''(x)=\frac{3}{8x^{5/2}}$ on [4, 5.1] is $f'''(4)=0.0117\ldots$

   so $|\text{Error bound}|\le\left|\frac{0.0117...(5.1-4)^3}{3!}\right|=0.003$.

   (d) 0.002

3. The series has terms that are alternating in sign, decreasing in magnitude, and having a limit of 0 so the error is less than the first truncated term by the Alternating Series Remainder.

   $|\text{Error}|<$ 6th term so $|\text{Error}|<\frac{5}{6!}$ or 0.007.

4. (a) $P(x)=\frac{\sqrt{3}}{2}-\frac{3x}{2}-\frac{9\sqrt{3}x^2}{2\cdot 2!}+\frac{27x^3}{2\cdot 3!}+\frac{81\sqrt{3}x^4}{2\cdot 4!}$

   (b)

$$|R_4(x)|=\left|\frac{f^{(5)}(z)(x-0)^5}{5!}\right|\le\left|\frac{243x^5}{5!}\right| \text{ so } \left|R_4\left(\frac{1}{6}\right)\right|\le\left(\frac{243}{5!}\right)\cdot\left(\frac{1}{6}\right)^5=\frac{1}{5!2^5}=\frac{1}{(120)(32)}<\frac{1}{3000}$$

5. The series has terms that are alternating in sign, decreasing in magnitude, and having a limit of 0 so the error is less than the first truncated term by the Alternating Series Remainder.

   $\int_0^1 e^{-x^2}dx\approx 1-\frac{1}{3}+\frac{1}{5(2!)}-\frac{1}{7(3!)}=\frac{26}{35}$. $|\text{Error}|<\frac{1}{216}<\frac{1}{200}$.

6. 0.003

7. (a) $\frac{1}{2}-\frac{x-5}{2^1(3)}+\frac{(x-5)^2}{2^2(4)}-\frac{(x-5)^3}{2^3(5)}$

   Since the series has terms that are alternating, decreasing in magnitude, and having a limit of 0. the error involved in approximating $f(6)$ with the third-degree Taylor polynomial is less than the fourth-degree term so

   $|\text{Error}|<\frac{(6-5)^4}{2^4(6)}=\frac{1}{96}<\frac{1}{50}$ by the Alternating Series Remainder.