Worksheet 9.5—Lagrange Error Bound
Show all work. Calculator permitted except unless specifically stated.

Free Response & Short Answer

1. (a) Find the fourth-degree Taylor polynomial for $\cos x$ about $x = 0$. Then use your polynomial to approximate the value of $\cos 0.8$, and use Taylor's Theorem to determine the accuracy of the approximation. Give three decimal places.

(b) Find the interval $[a, b]$ such that $a \le \cos 0.8 \le b$.

(c) Could $\cos 0.8$ equal 0.695? Show why or why not.

Answers

① (a) $P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \approx \cos x = f(x)$ $\quad c=0,\ x=0.8$

$$\cos 0.8 \approx P_4(0.8) = 1 - \frac{(0.8)^2}{2!} + \frac{(0.8)^4}{4!} = 0.697 = A$$

$$R_4(0.8) = \left| \frac{f^{5}(z)}{5!}(0.8-0)^5 \right| \leq \left| \frac{1}{5!}(0.8)^5 \right| = 0.0027306667 = B$$

* $f^{(5)}(z)$ has a max value of one on the interval $[0, 0.8]$ since one is the amplitude of $\cos x$ and its derivatives

** The Lagrange error here is also the Alternating series error

(b) $\cos 0.8 \in [A-B, A+B] = [0.694336, 0.699797] = I$

* $\cos 0.8$ actually equals $0.6967067093 \in I$

(c) $\cos 0.8$ <u>could</u> equal 0.695 because $0.695 \in I$ from part (b).

2. (a) Write a fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate e^{-1}, and find a Lagrange error bound for the maximum error when $|x| \le 1$. Give three decimal places.

(b) Find an interval $[a, b]$ such that $a \le e^{-1} \le b$.

Answers

② $f(x) = e^x \approx T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$, $c = 0$, $x = -1$

(a) $f(-1) = e^{-1} \approx T_4(-1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = 0.375 = A$

$R_4(-1) = \left| \frac{f^5(z)}{5!} (0 - (-1))^5 \right| \leq \left| \frac{e}{5!} \right| = 0.0226523486 = B$

* $f^{(5)}(z)$ has a max value of e^1 on $|x| \leq 1 \Rightarrow -1 \leq x \leq 1$

(b) $e^{-1} \in [A - B, A + B] = [0.352347, 0.397652] = I$

* e^{-1} actually equals $0.3678794412 \in I$

3. Let f be a function that has derivatives of all orders for all real numbers x. Assume that $f(5)=6$, $f'(5)=8$, $f''(5)=30$, $f'''(5)=48$, and $\left|f^{(4)}(x)\right| \le 75$ for all x in the interval $[5, 5.2]$.

(a) Find the third-degree Taylor polynomial about $x=5$ for $f(x)$.

(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.

(c) Find an interval $[a,b]$ such that $a \le f(5.2) \le b$. Give three decimal places.

(d) Could $f(5.2)$ equal 8.254? Show why or why not.

Answers

③ $f(5)=6,\ f'(5)=8,\ f''(5)=30,\ f'''(5)=48,\ \left|f^{(4)}(x)\right| \le 75 \ \forall x \in [5, 5.2]$

(a) $T_3(x) = 6 + 8(x-5) + \frac{30}{2!}(x-5)^2 + \frac{48}{3!}(x-5)^3 \approx f(x)$

(b) $f(5.2) \approx T_3(5.2) = 6 + 8(0.2) + 15(0.2)^2 + 8(0.2)^3 = \boxed{8.264} = A$

$R_3(5.2) = \left|\frac{f^{(4)}(z)}{4!}(5.2-5)^4\right| \le \left|\frac{75}{4!}(0.2)^4\right| = \boxed{0.005} = B$

(c) $f(5.2) \in [A-B, A+B] = [8.259, 8.269] = I$

(d) $f(5.2)$ could <u>not</u> equal 8.254 because $8.254 \notin I$ from part (c).

Review (Problems 4 - 7):

4. Find the first four nonzero terms of the power series for $f(x) = \sin x$ centered at $x = \frac{3\pi}{4}$.

5. Find the first four nonzero terms and the general term for the Maclaurin series for

(a) $f(x) = x\cos\left(x^3\right)$

(b) $g(x) = \frac{1}{1+x^2}$

6. Find the radius and interval of convergence for

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$

(b) $\sum_{n=0}^{\infty} (2n)!(x-5)^n$

Answers

(4) $f(x)=\sin x$, $c=\frac{3\pi}{4}$

$f(x)=\sin x$, $f(\frac{3\pi}{4})=\frac{\sqrt{2}}{2}$

$f'(x)=\cos x$, $f'(\frac{3\pi}{4})=-\frac{\sqrt{2}}{2}$

$f''(x)=-\sin x$, $f''(\frac{3\pi}{4})=-\frac{\sqrt{2}}{2}$

$f'''(x)=-\cos x$, $f'''(\frac{3\pi}{4})=\frac{\sqrt{2}}{2}$

$f^{(4)}(x)=\sin x$, $f^{(4)}(\frac{3\pi}{4})=\frac{\sqrt{2}}{2}$

So $f(x)=\sin x=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}(x-\frac{3\pi}{4})-\frac{\sqrt{2}/2}{2!}(x-\frac{3\pi}{4})^2+\frac{\sqrt{2}/2}{3!}(x-\frac{3\pi}{4})^3+\frac{\sqrt{2}/2}{4!}(x-\frac{3\pi}{4})^4+\cdots$

(5) (a) $f(x)=x\cos(x^3)$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$

$\cos(x^3)=1-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\frac{x^{18}}{6!}+\cdots+\frac{(-1)^n x^{6n}}{(2n)!}+\cdots$

$f(x)=x\cos(x^3)=x-\frac{x^7}{2!}+\frac{x^{13}}{4!}-\frac{x^{19}}{6!}+\cdots+\frac{(-1)^n x^{6n+1}}{(2n)!}+\cdots$

(b) $f(x)=\frac{1}{1+x^2}$

$$\begin{array}{r|l} & 1-x^2+x^4-x^6+\cdots \\ 1+x^2 & 1 \\ & \underline{1+x^2} \\ & -x^2 \\ & \underline{-x^2-x^4} \\ & x^4 \end{array}$$

So $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+\cdots$

(6) Radius and Interval of Convergence

(a) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{3^n n^2}$, $\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{3^{n+1}\cdot(n+1)^2}\cdot\frac{3^n\cdot n^2}{(x-2)^n}\right|$

$=\lim_{n\to\infty}\left|\frac{(x-2)n^2}{3(n+1)^2}\right|=\frac{1}{3}|x-2|<1$

so $|x-2|<3$, center $c=2$

Radius = 3, Interval $[(-1,5)]$

Test endpts:

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-3)^n}{3^n n^2}=\sum_{n=0}^{\infty}\frac{(-1)^{2n}3^n}{3^n n^2}=\sum_{n=0}^{\infty}\frac{1}{n^2}$ → convergent p-series

$x=5$: $\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{3^n n^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ → convergent alt series

So Interval is $[-1,5]$

(b) $\sum_{n=0}^{\infty}(2n)!(x-5)^n$

$\lim_{n\to\infty}\left|\frac{(2n+2)!(x-5)^{n+1}}{(2n)!(x-5)^n}\right|$

$=\lim_{n\to\infty}|(2n+2)(2n+1)(x-5)|$

$=\infty \not< 1$ so

Radius = 0

This series converges only at $x=5$, its center

7. Use the Maclaurin series for $\cos x$ to find $\lim_{x\to 0}\frac{1-\cos x}{x}$.

8. The Taylor series about $x=3$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=3$ is given by

$$f^{(n)}(3)=\frac{(-1)^n\, n!}{5^n\,(n+3)} \text{ and } f(3)=\frac{1}{3}$$

(a) Write the fourth-degree Taylor polynomial for f about $x=3$.

(b) Find the radius of convergence of the Taylor series for f about $x=3$.

(c) Show that the third-degree Taylor polynomial approximates $f(4)$ with an error less than $\frac{1}{4000}$.

Answers

(7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots\right)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1} x^{2n}}{(2n)!} + \cdots}{x} = 0/0$$

$$= \lim_{x\to 0} \frac{\cancel{x}\left(\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n)!} + \cdots\right)}{\cancel{x}} = \boxed{0}$$

(8) $f^{(n)}(3) = \frac{(-1)^n n!}{5^n(n+3)}$, $\underline{c=3}$, $f(3) = \frac{1}{3}$, $f'(3) = \frac{-1}{5\cdot 4}$, $f''(3) = \frac{2}{25\cdot 5}$, $f'''(3) = \frac{-6}{125\cdot 6}$, $f^{(4)}(3) = \frac{4!}{5^4\cdot 7}$

(a) $T_4(x) = \frac{1}{3} - \frac{1}{20}(x-3) + \frac{2/125}{2!}(x-3)^2 - \frac{6/(6\cdot 125)}{3!}(x-3)^3 + \frac{4!/(5^4\cdot 7)}{4!}(x-3)^4 \approx f(x)$

$= \frac{1}{3} - \frac{1}{20}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{750}(x-3)^3 + \frac{1}{4375}(x-3)^4$

(b) the n^{th} term for the Taylor series for $f(x)$ is:

$\frac{f^{(n)}(3)}{n!} = \frac{(-1)^n n!}{n!\,5^n(n+3)} = \frac{(-1)^n}{5^n(n+3)}$. Radius: $\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{5^{n+1}(n+4)} \cdot \frac{5^n(n+3)}{(x-3)^n} \right|$

$\lim_{n\to\infty} |x-3| \frac{n+3}{5(n+4)} = \frac{1}{5}|x-3| < 1$ so $|x-3| < 5$ and $\boxed{\text{Radius} = 5}$

(c) $f(4) \approx T_4(4)$. $R_4(4) = \left| \frac{f^{(5)}(z)}{5!}(4-3)^5 \right| \le \left| \frac{5!}{5!\,5^5\cdot 8}(1)^5 \right| = \frac{1}{25000} = 0.00004 < \frac{1}{4000}$

$x=4,\ c=3$

* on the interval $[3,4]$, $f^{(5)}(z) \approx f^{(5)}(3) = \frac{(-1)5!}{5^5(8)}$

9. Let *f* be a function that has derivatives of all orders on the interval $(-1,1)$. Assume $f(0)=1$, $f'(0)=\frac{1}{2}$, $f''(0)=-\frac{1}{4}$, $f'''(0)=\frac{3}{8}$, and $\left|f^{(4)}(x)\right| \le 6$ for all x in the interval $(-1,1)$.

(a) Find the third-degree Taylor polynomial about $x=0$ for the function *f*.

(b) Use your answer to part (a) to estimate the value of $f(0.5)$.

(d) What is the maximum possible error for the approximation made in part (b)?

Answers

(9) $f(0)=1,\ f'(0)=\frac{1}{2},\ f''(0)=-\frac{1}{4},\ f'''(0)=\frac{3}{8},\ |f^{(4)}(x)|\le 6\ \forall x\in(-1,1)$, $\underline{c=0}$

(a) $T_3(x) = 1 - \frac{1}{2}x - \frac{1/4}{2!}x^2 + \frac{3/8}{3!}x^3 \approx f(x)$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

(b) $f(0.5) \approx T_3(0.5) = 0.7265625 = \frac{93}{128}$

(c) $R_3(0.5) = \left|\frac{f^{(4)}(z)}{4!}(0.5-0)^4\right| \le \left|\frac{6}{4!}\left(\frac{1}{2}\right)^4\right| = \underline{0.015625} = \underline{\frac{1}{64}}$

$c=0,\ x=0.5$ ← max possible error

10. Let f be the function defined by $f(x) = \sqrt{x}$.

(a) Find the second-degree Taylor polynomial about $x = 4$ for the function f.

(b) Use your answer to part (a) to estimate the value of $f(4.2)$.

(c) Find a bound on the error for the approximation in part (b).

Answers

(10) $f(x) = \sqrt{x}$

(a) $c = 4$

$f(x) = x^{1/2}$, $f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$, $f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$, $f''(x) = -\frac{1}{32}$

So $f(x) = \sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2$

$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

(b) $f(4.2) = \sqrt{4.2} \approx T_2(4.2) = 2 + \frac{1}{4}(0.2) - \frac{1}{64}(0.2)^2 = 2.049375 = A$

(c) $T_2(4.2) = \left|\frac{f'''(z)}{3!}(4.2-4)^3\right| \leq \left|\frac{3}{3!(256)}(0.2)^3\right| = \frac{1}{(250)(256)} = 0.0000156Z$

or $\frac{1}{64000}$ ← MAX ERROR

* $f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}}$. $f'''(x)$ has its max value on $[4, 4.2]$ at $x = 4$, so $f'''(z) = \frac{3}{8\sqrt{4^5}} = \frac{3}{256}$

11. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ for all x for which the series converges.

(a) Find the interval of convergence of this series.

(b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.

(c) Estimate the error involved in the approximation in part (b). Show your reasoning.

Answers

(11) $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ (a) Interval of Convergence: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{1}{2}x\right|=\frac{1}{2}|x|<1$

$|x-0|<2$

possible IOC: $[(-2,2)]$

Test endpts: $x=-2$: $\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n \to$ Diverges

Radius = 2

$x=2$: $\sum_{n=0}^{\infty}\frac{(2^n)}{2^n}=\sum_{n=0}^{\infty}1 \to$ Diverges

So Interval of Convergence is $\boxed{(-2,2)}$

(b) $\sum_{n=0}^{\infty}\frac{x^n}{2^n}=1+\frac{x}{2}+\frac{x^2}{4}$

$f(x)\approx 1+\frac{x}{2}+\frac{x^2}{4}=T_2(x)$

$f(-\frac{1}{2})\approx T_2(-\frac{1}{2})=1-\frac{1}{4}+\frac{1}{16}=\boxed{0.8125=\frac{13}{16}}$

(c) For $x=-\frac{1}{2}$, the series is an alternating series, so the maximum error will be the magnitude of the first unused term in the series for $f(-\frac{1}{2})=1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots+\frac{(-1)^n}{4^n}+\cdots$

($-\frac{1}{64}$ ← 1st unused term in $T_2(-\frac{1}{2})$)

So error $\leq\left|-\frac{1}{64}\right|=\boxed{\frac{1}{64}}$

12. Let f be the function given by $f(x) = \cos\left(3x + \frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right) \right| < \frac{1}{3000}$.

Answers

(12) $f(x) = \cos\left(3x + \frac{\pi}{6}\right) \quad c = 0$

$f(x) = \cos\left(3x + \frac{\pi}{6}\right),\ f(0) = \frac{\sqrt{3}}{2}$

$f'(x) = -3\sin\left(3x + \frac{\pi}{6}\right),\ f'(0) = -\frac{3}{2}$

$f''(x) = -9\cos\left(3x + \frac{\pi}{6}\right),\ f''(0) = -\frac{9\sqrt{3}}{2}$

$f'''(x) = 27\sin\left(3x + \frac{\pi}{6}\right),\ f'''(0) = \frac{27}{2}$

$f^{(4)}(x) = 81\cos\left(3x + \frac{\pi}{6}\right),\ f^{(4)}(0) = \frac{81\sqrt{3}}{2}$

$f^{(5)}(x) = -243\sin\left(3x + \frac{\pi}{6}\right),\ f^{(5)}(0) = -\frac{243}{2}$

(a) $P_4(x) = \frac{\sqrt{3}}{2} - \frac{3}{2}x - \frac{9\sqrt{3}/2}{2!}x^2 + \frac{27/2}{3!}x^3 + \frac{81\sqrt{3}/2}{4!}x^4 \approx f(x)$

$P_4(x) = \frac{\sqrt{3}}{2} - \frac{3}{2}x - \frac{9\sqrt{3}}{4}x^2 + \frac{9}{4}x^3 + \frac{27\sqrt{3}}{16}x^4 \approx f(x)$

(b) $R_4\left(\frac{1}{6}\right) = \left|\frac{f^{(5)}(z)}{5!}\left(\frac{1}{6} - 0\right)^5\right| \le \left\|\frac{243}{5!}\left(\frac{1}{6}\right)^5\right| = \left(\frac{81}{40}\right)\left(\frac{1}{7776}\right) \approx 0.0002604 = A$

$c = 0, x = \frac{1}{6}$ * the max value of $|f^{(5)}(z)|$ is 243, the amplitude of $f^{(5)}(x)$

$\frac{1}{3000} \approx 0.0003333 = B$ $\quad$ $A < B$, where $A = \left|f\left(\frac{1}{6}\right) - P_4\left(\frac{1}{6}\right)\right|$

13. (Review) Use series to find an estimate for $I=\int_0^1 e^{-x^2}\,dx$ that is within 0.001 of the actual value.

Justify.

14. The Taylor series about $x=5$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence. The *n*th derivative of *f* at $x=5$ is given by

$$f^{(n)}(5)=\frac{(-1)^n\, n!}{2^n\,(n+2)} \text{ and } f(5)=\frac{1}{2}.$$

Show that the sixth-degree Taylor polynomial for *f* about $x=5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

Answers

(13) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$

$I = \int_0^1 e^{-x^2}\,dx = x - \frac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + \cdots \Big|_0^1$

$= \left(1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} + \cdots + \frac{(-1)^n}{(2n+1)n!} + \cdots\right) - (0) = I$

* the approximation for I must be within $\frac{1}{1000}$ of the actual value.
* I is an alternating series, so error is less than the magnitude of the 1st unused term. The 1st term that is less than 0.001 is $\left|\frac{1}{11\cdot 5!}\right| \approx 0.000757$, so $I \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}$ $\approx \boxed{0.747486} = A$

*P.S. $\int_0^1 e^{-x^2}dx = 0.7468241328 = B$, so $|A - B| = 0.000006626 < 0.001$ ← actual error

(14) $c = 5$, $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$, $f(5) = \frac{1}{2}$

$f(6) \approx T_6(6)$, $R_6(6) = \left|\frac{f^{(7)}(z)}{7!}(6-5)^7\right| \le \left|\frac{5!}{7!\,2^5\cdot 7}\right| = \boxed{\frac{1}{(42)(32)(7)} = A}$

* $\left|f^{(7)}(z)\right|$ is approximated by $\left|f^{(7)}(5)\right| = \left|\frac{(-1)5!}{2^5(5+2)}\right| = \frac{5!}{2^5(7)}$

$\boxed{A = \frac{1}{9408} < \frac{1}{1000}}$ **notice the question did not ask us to approximate $f(6)$.

Multiple Choice

15. Suppose a function *f* is approximated with a fourth-degree Taylor polynomial about $x = 1$. If the maximum value of the fifth derivative between $x = 1$ and $x = 3$ is 0.01, that is, $\left|f^{(5)}(x)\right| < 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

16. What are all the values of *x* for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x < 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$ (E) All real x

Answers

(15) $f(3) \approx T_4(3)$, $x=3, c=1$, $|f^{(5)}(x)| < 0.01$

$$R_4(3) = \left| \frac{f^{(5)}(z)}{5!}(3-1)^5 \right| \leq \left| \frac{0.01}{5!}(2^5) \right| = \boxed{0.0026666} \quad \boxed{E}$$

(16) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$; Interval of Convergence: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1 \ \forall x$

So Interval is $(-\infty, \infty)$ and radius is ∞. $\boxed{E}$
& all real x

17. The coefficient of x^6 in the Taylor series expansion about $x=0$ for $f(x)=\sin\left(x^2\right)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

18. The maximum error incurred by approximating the sum of the series $1-\frac{1}{2!}+\frac{2}{3!}-\frac{3}{4!}+\frac{4}{5!}-\cdots$ by the sum of the first six terms is

(A) 0.001190 (B) 0.006944 (C) 0.33333 (D) 0.125000 (E) None of these

Answers

(17) The coefficient of x^6 for $c=0$, $f(x)=\sin(x^2)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$

the coeff of x^6 is $-\frac{1}{3!} = \boxed{-\frac{1}{6}}$ $\boxed{A}$

(18) $\underbrace{1 - \frac{1}{2!} + \frac{2}{3!} - \frac{3}{4!} + \frac{4}{5!} - \frac{5}{6!}}_{\text{1st 6 terms}} + \underbrace{\frac{6}{7!}}_{\text{1st unused term}}$ Maximum error $\leq \left|\frac{6}{7!}\right| = 0.0011904762$

$\boxed{A}$

19. If f is a function such that $f'(x)=\sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x=0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

20. Now that you have finished the last question of the last "new concept" worksheet of your high school career, how do you feel? (Show your work)

(A) Relieved (B) Very Sad (C) Euphoric (D) Tired (E) All of these

Answers

(19) $f'(x) = \sin(x^2)$, $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = f'(x)$,

$f(x) = \int f'(x)\,dx = C + \frac{1}{2}x^2 - \frac{1}{7\cdot 3!}x^7 + \frac{1}{11\cdot 5!}x^{11} + \cdots$; coeff of x^7 is $-\frac{1}{7\cdot 3!} = \boxed{-\frac{1}{42}}$ $\boxed{D}$

(20) (A), (B), or (C) but not (D)