LAGRANGE ERROR BOUND
ADDITIONAL PRACTICE:

1 The hyperbolic sine is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$. A third-order Taylor polynomial approximation is $\sinh x \approx x + \frac{x^3}{3!}$. If this is used to approximate $\sinh x$ for $|x| \le 2$, which is the LaGrange error bound?

(A) 4.836 (B) 3.627 (C) 2.718 (D) 2.508 (E) 2.418

2 What is the smallest order of Taylor polynomial centered at $x = 1$ which will approximate e^{x-1} on the domain $-1 \le x \le 3$ with LaGrange error bound less than 1?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

3 The approximation $\ln(1+x) \approx x - \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation Theorem to get a bound for the maximum error when $|x| \le 0.1$.

4 The approximation $e^x \approx 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| \le 0.1$.

5 Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

(a) Find the radius of convergence of this series.

(b) Use the first three terms of this series to approximate $f\left(\frac{-1}{3}\right)$.

(c) Estimate the error involved in the approximation in part (b).

Answers

SOLUTIONS:

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$f(x)=\frac{e^x-e^{-x}}{2} \Rightarrow f'(x)=\frac{e^x+e^{-x}}{2} \Rightarrow f''(x)=\frac{e^x-e^{-x}}{2} \Rightarrow f'''(x)=\frac{e^x+e^{-x}}{2} \Rightarrow f^{iv}(x)=\boxed{\frac{e^x-e^{-x}}{2}}$

$R_3(x)\le\left|\frac{e^x-e^{-x}}{2}\bullet\frac{x^4}{4!}\right| \Rightarrow R_3(2)\le\left|\frac{e^2-e^{-2}}{2}\bullet\frac{2^4}{4!}\right| \Rightarrow R_3(2)\le\left|\frac{e^2-e^{-2}}{2}\bullet\frac{2^4}{4!}\right| \Rightarrow R_3(2)\le\boxed{\boxed{2.41791}} \ \therefore \ \boxed{\boxed{\text{E}}}$

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$e^x\approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots \Rightarrow$

$e^{x-1}\approx 1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\frac{(x-1)^5}{5!}+\ldots$

$\text{ERROR}\le\left|f^{n+1}\bullet\frac{(x-1)^{n+1}}{(n+1)!}\right| \Rightarrow \left[\begin{array}{l}\text{ERROR biggest when } x=3\\ \text{Deriv. of } e^{x-1} \text{ is always } e^{x-1}\end{array}\right] \Rightarrow$

$\text{ERROR}\le\left|e^{3-1}\bullet\frac{(3-1)^{n+1}}{(n+1)!}\right| \Rightarrow \text{ERROR}\le\left|e^2\bullet\frac{2^{n+1}}{(n+1)!}\right|$

$e^2\bullet\frac{2^{5+1}}{(5+1)!}\approx 0.6568<1 \quad \therefore \quad \boxed{\boxed{n=5}}$

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<table>
<tr><th>n</th><th>error</th></tr>
<tr><td>1</td><td>$e^2\bullet\frac{2^{1+1}}{(1+1)!}\approx 14.778$</td></tr>
<tr><td>2</td><td>$e^2\bullet\frac{2^{2+1}}{(2+1)!}\approx 9.852$</td></tr>
<tr><td>3</td><td>$e^2\bullet\frac{2^{3+1}}{(3+1)!}\approx 4.926$</td></tr>
<tr><td>4</td><td>$e^2\bullet\frac{2^{4+1}}{(4+1)!}\approx 1.970$</td></tr>
<tr><td>5</td><td>$e^2\bullet\frac{2^{5+1}}{(5+1)!}\approx 0.6568$</td></tr>
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$f(x)=\ln(1+x) \Rightarrow f'(x)=(1+x)^{-1} \Rightarrow$

$f''(x)=-1(1+x)^{-2} \Rightarrow f'''(x)=2(1+x)^{-3}$

$\text{Error}\le\left|\frac{2}{(1+x)^3}\bullet\frac{x^3}{3!}\right|$

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<table>
<tr><th>x</th><th>Error/Remainder</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.1</td><td>$\frac{2}{(1+x)^3}\bullet\frac{x^3}{3!}=\frac{2}{1.1^3}\bullet\frac{0.1^3}{6}\approx 0.0002504$</td></tr>
<tr><td>-0.1</td><td>$\left|\frac{2}{(1+x)^3}\bullet\frac{x^3}{3!}\right|=\left|\frac{2}{0.9^3}\bullet\frac{-0.1^3}{6}\right|\approx\boxed{\boxed{0.000457}}$</td></tr>
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$\text{ERROR}\le\left|e^x\bullet\frac{x^n}{n!}\right|=\left|e^x\bullet\frac{x^3}{3!}\right|=e^{0.1}\bullet\frac{(0.1)^3}{3!}=\boxed{\boxed{1.842 \text{ X } 10^{-4}}}$

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(a) $\lim\limits_{n\to\infty}\left|\frac{x^{n+1}(n+1)^{n+1}}{(n+1)!}\bullet\frac{n!}{x^n n^n}\right| \Rightarrow \lim\limits_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\bullet\frac{(n+1)^{n+1}}{n^n}\bullet\frac{n!}{(n+1)!}\right|\le 1 \Rightarrow \lim\limits_{n\to\infty}\left|x\bullet\frac{(n+1)^{n+1}}{n^n}\bullet\frac{1}{n+1}\right|\le 1 \Rightarrow$

$\lim\limits_{n\to\infty}\frac{(n+1)^n}{n^n}|x|\le 1 \Rightarrow \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n|x|\le 1 \Rightarrow e|x|\le 1 \Rightarrow |x|\le\frac{1}{e} \quad \therefore \quad \text{ROC}=\boxed{\boxed{\frac{1}{e}}}$

(b) $f(x)\approx x+\frac{4x^2}{2}+\frac{27x^3}{6} \Rightarrow f\left(\frac{-1}{3}\right)\approx\frac{-1}{3}+\frac{4(1/9)}{2}-\frac{27(1/27)}{6}=\boxed{\boxed{\frac{-5}{18}}}$

(c) Alternating Series, So Error $\le\left|\frac{x^4\bullet 4^4}{4!}\right|=\frac{1}{81}\bullet\frac{256}{24}=\boxed{\boxed{\frac{32}{243}}}$

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