# Notes: 8.2 Geometric Series

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| Infinite Series | $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$<br>end<br>start |
| Convergence/ Divergence of Series | If the sequence of **partial sums** $\{S_n\}$ converges to $S$, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit $S$ is called the sum of the series ($S = a_1 + a_2 + \cdots + a_n + \cdots$). If $\{S_n\}$ diverges, then the series diverges. |
| Using Partial Sums to Determine Convergence/ Divergence (Arithmetic) | $$\sum_{n=1}^{\infty}(2n+1) = 3 + 5 + 7 + \cdots$$<br>Find the first five terms of the sequence of partial sums and list them below.<br>$\to 3, 8, 15, 24, 35, \ldots$<br>$S_1 = 3$<br>$S_2 = 3+5=8$<br>$S_3 = 3+5+7=15$<br>$S_4 = 3+5+7+9=24$<br>$S_5 = 3+5+7+9+11=35$<br>What do you notice about the results? Do you think that the sequence of partial sums has a limit or a bound?<br>partial sums increase without bound<br>$\Rightarrow$ sequence of partial sums diverges<br>$\Rightarrow \sum_{n=1}^{\infty}(2n+1)$ diverges<br>What about other arithmetic series? Do you think that there are any arithmetic series that converge? Why or why not?<br>all will diverge $\to$ sequence of partial sums will increase or decrease without bounds |

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| Using Partial Sums to Determine Convergence/ Divergence (Geometric) | $$\sum_{n=1}^{\infty}\frac{3}{2^n}=\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+\frac{3}{64}+\frac{3}{128}+\frac{3}{256}+\frac{3}{512}+\frac{3}{1024}+\cdots$$ Find the first ten terms of the sequence of partial sums, and list them below.<br>$S_1=1.5$ $S_5=2.906$<br>$S_2=2.25$ $S_6=2.953$ $S_9=2.994$<br>$S_3=2.625$ $S_7=2.977$ $S_{10}=2.997$<br>$S_4=2.812$ $S_8=2.988$<br>What do you notice about the results? Do you think the sequence of partial sums has a limit or a bound?<br>appears to approach 3<br>$\sum_{n=1}^{\infty}\frac{3}{2^n}$ converges to 3<br>What about other geometric series? Do you think that they all converge?<br>some will |
| Error | The error in an approximation is the difference between the actual sum of the infinite series and the sum of the terms that you have added. What is the error in your approximation. |
| Special types of series: Telescoping | $\lvert Error\rvert = \lvert Actual\ value - Approximation\rvert =$ $\lvert 3-2.997\rvert = 0.003$<br>How could you make the error smaller? more terms |
| Using Partial Sums to Determine Convergence/ Divergence (Geometric) | $$\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n=\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\cdots$$ Find the first five terms of the sequence of partial sums and list them below.<br>$S_1=1.5$ $S_4=12.188$<br>$S_2=3.75$ $S_5=19.781$<br>$S_3=7.125$<br>What do you notice about the results? Do you think that the sequence of partial sums has a limit or a bound?<br>sequence of partial sums increases without bound $\Rightarrow \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ diverges |

**Sums of Geometric Series**

Formula to help find the sum of a geometric series:

$$S = \frac{a}{1-r} = \frac{\text{first term}}{1 - \text{ratio}}$$

What has to be true for this formula to give you a valid answer?

infinite number of terms

$|r| < 1$

**Geometric Series Test**

A geometric series is in the form $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=1}^{\infty} ar^{n+1}$, $a \neq 0$, where $a$ is the first term of the series and $r$ is the common ratio.

The geometric series **diverges** if $|r| \geq 1$.

The geometric series **converges** if $|r| < 1$ and the sum is $S = \frac{a}{1-r}$

**Examples for Geometric Series Test**

Determine whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n$$

$r = \frac{1}{2}$

geometric $|r| < 1$

converges by geometric series test

$$S = \frac{\frac{3}{2}}{1 - \frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \qquad r = \frac{3}{2}$$

$|r| \geq 1$

diverges by geometric series test

Consider the geometric series $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$ for all $n$. The first term of the series is $a_1 = 108$, and the third term is $a_3 = 12$. What is the value of $\sum_{n=1}^{\infty} a_n$?

sum = ?

$a_1 = 108$

$\cdot\frac{1}{3}$ $a_2 = 36$

$\cdot\frac{1}{3}$ $a_3 = 12$

$\frac{1}{9}$

$$S = \frac{108}{1 - \frac{1}{3}} = 162$$

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| Practice Problems | Find the sum, if it exists, for the following series. |

(a) The first term of an infinite geometric sequence is 3, and the common ratio is $\frac{2}{5}$.

$5$

(b) $\frac{\sqrt{2}}{5}+\frac{1}{5}+\frac{\sqrt{2}}{10}+\frac{1}{10}+\cdots$

$$\frac{\frac{\sqrt{2}}{5}}{1-\frac{1}{\sqrt{2}}}$$

(c) $\sum_{i=1}^{\infty} 8\left(\frac{5}{4}\right)^{i}$

diverges

(d) $\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+2}} = \sum_{n=1}^{\infty} \frac{(2^2)^n}{5^n \cdot 5^2} = \sum_{n=1}^{\infty} \frac{4^n}{25 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{25}\left(\frac{4}{5}\right)^n$

$$S = \frac{\frac{4}{125}}{1-\frac{4}{5}} = \frac{4}{25}$$

Find the following information regarding different infinite geometric series.

(a) The sum of an infinite geometric series is 81 and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

$27, 18, 12$

(b) The sum of an infinite geometric series is 125 and its $7^{th}$ and $8^{th}$ terms are $\frac{192}{625}$ and $\frac{384}{3125}$, respectively. Find the first three terms of the series

$75, 30, 12$

(c) The first term of an infinite geometric series is $-8$, and its sum is $-\frac{64}{3}$. Find the first four terms of the series.

$-8, -5, -\frac{25}{8}, -\frac{125}{64}$