Telescoping Series Test.

A series such as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is called a **telescoping series** because it collapses to one term or just a few terms. If a series collapses to a finite sum, then it converges by the **Telescoping Series Test**.

Example 10:
Determine whether the following series converges or diverges. If they converge, find their sum.

SUM(SEQ($\frac{1}{2n+1}-\frac{1}{2n+3}$), n, 1, 500)

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$

$= \left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{7}}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$

$\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n+3}\right)$

$\frac{1}{3}-0 \rightarrow$

IT CONVERGES TO $\frac{1}{3}$

(b) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

$\frac{1}{n}-\frac{1}{n+1}$

$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

EVERYTHING CANCELS OUT EXCEPT FOR 1 AND $\frac{1}{n-1}$

$\therefore 1-\frac{1}{n-1}=1-0$

SO IT CONVERGES TO 1.

(c) $\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)}$

~~$\frac{1}{n+1}-\frac{1}{n+3}$~~

~~$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=\left(\frac{1}{2}-\frac{1}{4}\right)+$~~

PARTIAL FRACTIONS

$\sum_{n=1}^{\infty}\left(\frac{1/2}{n+1}-\frac{1/2}{n+3}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$

$=\frac{1}{2}\left[\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+3}\right)\right]$

$=\frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right]$

$\lim_{n\to\infty}\frac{1}{2}\left(\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right)=\frac{5}{12}$

$\therefore$ IT CONVERGES TO $\frac{5}{12}$