Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Sample Problem #3: **APPLYING THE STRATEGIES FOR TESTING SERIES**

Determine the convergence or divergence of the series:

a) $\sum_{n=1}^{\infty} \frac{n-1}{4n+1}$

$a_n = \frac{n-1}{4n+1}$

$\underbrace{\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{4n+1} = \frac{1}{4} \neq 0}$

$\boxed{\sum_{n=1}^{\infty} \frac{n-1}{4n+1} \text{ is divergent}}$

b) $\sum_{n=1}^{\infty}\left(\frac{5}{17}\right)^n = \frac{5}{17} + \frac{5}{17}\cdot\left(\frac{5}{17}\right)^1 + \frac{5}{17}\cdot\left(\frac{5}{17}\right)^2 + \cdots$

Geometric Series

$a = 5/17,\ r = 5/17$

$$\sum_{n=0}^{\infty} \frac{5}{17}\left(\frac{5}{17}\right)^n = \frac{5/17}{1-5/17} = \frac{5/17}{12/17} = \boxed{5/12}$$

$$\boxed{\sum_{n=1}^{\infty} \left(5/17\right)^n \text{ converges, and its sum is } 5/12}$$

c) $\sum_{n=1}^{\infty} n^2 e^{-n^3} = \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

$$f(x) = \frac{x^2}{e^{x^3}}$$

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{x^2}{e^{x^3}}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{3}\int_0^b \frac{3x^2}{e^{x^3}}\,dx \qquad u = x^3$$

$$= \lim_{b\to\infty} \int_0^{b^3} e^{-u}\,du$$

$$= \lim_{b\to\infty} \left. -\frac{1}{e^u} \right]_0^{b^3}$$

$$= \lim_{b\to\infty} -\frac{1}{e^{b^3}} + 1 = 1$$

$$\boxed{\int_1^{\infty} f(x)\,dx \text{ converges then } \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}} \text{ converges too.}}$$

d) $\sum_{n=1}^{\infty} \frac{1}{4n+1}$

Divergent

$a_n = \frac{1}{4n+1}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n}{4n+1} = 1$

$b_n = \frac{1}{4n}$

↳ divergent series

$f(x) = \frac{1}{4x}$, $\int_1^{\infty} \frac{1}{4x}\,dx = \lim_{b\to\infty} \frac{1}{4}(\ln b - \ln 1) \to \infty$

By the Limit Comparison Test, if b_n diverges then a_n diverges too.

e) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4n+1} = 0$

$a_{n+1} = \frac{1}{4(n+1)+1} = \frac{1}{4n+5} < \frac{1}{4n+1} = a_n$

By the Alternating Series Test

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n+1}$ converges

f) $\sum_{n=1}^{\infty} \frac{n!}{15^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{15^{n+1}}}{\frac{n!}{15^n}} = \frac{(n+1)\cancel{n!}}{15 \cdot \cancel{15^n}} \cdot \frac{\cancel{15^n}}{\cancel{n!}}$$

$$= \frac{n+1}{15}$$

$$\underbrace{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{15} = \infty}$$

$\boxed{\text{By the } \underline{\text{Ratio Test}} \ \sum_{n=1}^{\infty} \frac{n!}{15^n} \text{ diverges.}}$

g) $\sum_{n=1}^{\infty} \left(\frac{n+1}{4n+1} \right)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n+1}{4n+1}\right)^n} = \frac{n+1}{4n+1}$$

$$\underbrace{\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n+1}{4n+1} = 1/4 < 1}$$

$\boxed{\text{By the } \underline{\text{Root Test}} \ \sum_{n=1}^{\infty} \left(\frac{n+1}{4n+1}\right)^n \text{ converges.}}$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \dfrac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \dfrac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	