10.5 The Ratio and Root Tests

Preliminary Questions

1. In the Ratio Test, is ρ equal to $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$?

SOLUTION In the Ratio Test ρ is the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

2. Is the Ratio Test conclusive for $\sum_{n=1}^{\infty}\frac{1}{2^n}$? Is it conclusive for $\sum_{n=1}^{\infty}\frac{1}{n}$?

SOLUTION The general term of $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is $a_n = \frac{1}{2^n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}}\cdot\frac{2^n}{1} = \frac{1}{2},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1.$$

Consequently, the Ratio Test guarantees that the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges.

The general term of $\sum_{n=1}^{\infty}\frac{1}{n}$ is $a_n = \frac{1}{n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1}\cdot\frac{n}{1} = \frac{n}{n+1},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

The Ratio Test is therefore inconclusive for the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

3. Can the Ratio Test be used to show convergence if the series is only conditionally convergent?

SOLUTION

No. The Ratio Test can only establish absolute convergence and divergence, not conditional convergence.

Exercises

In Exercises 1–20, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

SOLUTION With $a_n = \frac{1}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} = \frac{1}{5} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges by the Ratio Test.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{5^n}$

SOLUTION With $a_n = \frac{(-1)^{n-1}n}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n} = \frac{n+1}{5n} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{5^n}$ converges by the Ratio Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^{n+1}} \cdot \frac{n^n}{1} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot \frac{1}{e} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Ratio Test.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$

SOLUTION With $a_n = \frac{3n+2}{5n^3+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3(n+1)+2}{5(n+1)^3+1} \cdot \frac{5n^3+1}{3n+2} = \frac{3n+5}{3n+2} \cdot \frac{5n^3+1}{5(n+1)^3+1},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1 = 1.$$

Therefore, for the series $\displaystyle\sum_{n=0}^{\infty} \frac{3n+2}{5n^3+1}$, the Ratio Test is inconclusive.

We can show that this series converges by using the Limit Comparison Test and comparing with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

SOLUTION With $a_n = \frac{n}{n^2+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+}$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot 1 = 1.$$

Therefore, for the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, the Ratio Test is inconclusive.

We can show that this series diverges by using the Limit Comparison Test and comparing with the divergent harmonic series.

6. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

SOLUTION With $a_n = \frac{2^n}{n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = \frac{2n}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2^n}{n}$ diverges by the Ratio Test.

7. $\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

SOLUTION With $a_n = \frac{2^n}{n^{100}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^{100}} \cdot \frac{n^{100}}{2^n} = 2\left(\frac{n}{n+1}\right)^{100} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot 1^{100} = 2 > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$ diverges by the Ratio Test.

8. $\sum_{n=1}^{\infty} \frac{n^3}{3^{n^2}}$

SOLUTION With $a_n = \frac{n^3}{3^{n^2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{(n+1)^2}} \cdot \frac{3^{n^2}}{n^3} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3^{2n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1^3 \cdot 0 = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{3^{n^2}}$ converges by the Ratio Test.

9. $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

SOLUTION With $a_n = \frac{10^n}{2^{n^2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{10^n} = 10 \cdot \frac{1}{2^{2n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 10 \cdot 0 = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ converges by the Ratio Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

SOLUTION With $a_n = \frac{e^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \frac{e}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$ converges by the Ratio Test.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

SOLUTION With $a_n = \frac{e^n}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{e^n} = \frac{e}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{e}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot \frac{1}{e} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges by the Ratio Test.

12. $\sum_{n=1}^{\infty} \frac{n^{40}}{n!}$

SOLUTION With $a_n = \frac{n^{40}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{40}}{(n+1)!} \cdot \frac{n!}{n^{40}} = \frac{1}{n+1}\left(\frac{n+1}{n}\right)^{40} = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{40},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot 1 = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^{40}}{n!}$ converges by the Ratio Test.

13. $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

SOLUTION With $a_n = \frac{n!}{6^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{6^{n+1}} \cdot \frac{6^n}{n!} = \frac{n+1}{6} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{n!}{6^n}$ diverges by the Ratio Test.

14. $\sum_{n=1}^{\infty} \frac{n!}{n^9}$

SOLUTION With $a_n = \frac{n!}{n^9}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^9} \cdot \frac{n^9}{n!} = \frac{n^9}{(n+1)^8} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{n^9}$ diverges by the Ratio Test.

15. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

SOLUTION With $a_n = \frac{1}{n \ln n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{1} = \frac{n}{n+1}\frac{\ln n}{\ln(n+1)},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}.$$

Now,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Thus, $\rho = 1$, and the Ratio Test is inconclusive for the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Using the Integral Test, we can show that the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

SOLUTION With $a_n = \frac{1}{(2n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(2n+2)!} \cdot \frac{(2n)!}{1} = \frac{1}{(2n+2)(2n+1)} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges by the Ratio Test.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

SOLUTION With $a_n = \frac{n^2}{(2n+1)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{n^2} = \left(\frac{n+1}{n}\right)^2 \frac{1}{(2n+3)(2n+2)},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1^2 \cdot 0 = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$ converges by the Ratio Test.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

SOLUTION With $a_n = \dfrac{(n!)^3}{(3n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{n^3+3n^2+3n+1}{27n^3+54n^2+33n+6}$$

$$= \frac{1+3n^{-1}+3n^{-2}+1n^{-3}}{27+54n^{-1}+33n^{-2}+6n^{-3}}$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{27} < 1$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ converges by the Ratio Test.

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n+1}$

SOLUTION With $a_n = \dfrac{1}{2^n+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}+1} \cdot \frac{2^n+1}{1} = \frac{1+2^{-n}}{2+2^{-n}}$$

and

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1$$

Therefore, the series $\sum_{n=2}^{\infty} \frac{1}{2^n + 1}$ converges by the Ratio Test.

20. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

SOLUTION With $a_n = \dfrac{1}{\ln n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\ln n} \cdot \frac{\ln(n+1)}{1} = \frac{\ln(n+1)}{\ln n}$$

and (using L'Hôpital's rule)

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x+1)}{\frac{d}{dx}\ln x} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

Therefore, the Ratio Test is inconclusive for $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$. This series can be shown to diverge using the Comparison Test with the harmonic series since $\ln n < n$ for $n \geq 2$.

21. Show that $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k.

SOLUTION With $a_n = n^k 3^{-n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^k 3^{-(n+1)}}{n^k 3^{-n}} = \frac{1}{3}\left(1+\frac{1}{n}\right)^k,$$

and, for all k,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k by the Ratio Test.

22. Show that $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$ converges if $|x| < 1$.

SOLUTION With $a_n = n^2 x^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2|x|^{n+1}}{n^2|x|^n} = \left(1+\frac{1}{n}\right)^2|x| \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1\cdot|x| = |x|.$$

Therefore, by the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} n^2 x^n$ converges provided $|x| < 1$.

23. Show that $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges if $|x| < \frac{1}{2}$.

SOLUTION With $a_n = 2^n x^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x|^{n+1}}{2^n|x|^n} = 2|x| \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x|.$$

Therefore, $\rho < 1$ and the series $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges by the Ratio Test provided $|x| < \frac{1}{2}$.

24. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ converges for all r.

SOLUTION With $a_n = \frac{r^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{(n+1)!} \cdot \frac{n!}{|r|^n} = \frac{|r|}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot |r| = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ converges by the Ratio Test for all r.

25. Show that $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$.

SOLUTION With $a_n = \frac{r^n}{n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{n+1} \cdot \frac{n}{|r|^n} = |r|\frac{n}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot |r| = |r|.$$

Therefore, by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges provided $|r| < 1$.

26. Is there any value of k such that $\sum_{n=1}^{\infty} \frac{2^n}{n^k}$ converges?

SOLUTION With $a_n = \frac{2^n}{n^k}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^k} \cdot \frac{n^k}{2^n} = 2\left(\frac{n}{n+1}\right)^k,$$

and, for all k,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot 1^k = 2 > 1.$$

Therefore, by the Ratio Test, there is no value for k such that the series $\sum_{n=1}^{\infty} \frac{2^n}{n^k}$ converges.

27. Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. *Hint:* Use $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

SOLUTION With $a_n = \frac{n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Ratio Test.

In Exercises 28–33, assume that $|a_{n+1}/a_n|$ converges to $\rho = \frac{1}{3}$. What can you say about the convergence of the given series?

28. $\sum_{n=1}^{\infty} na_n$

SOLUTION Let $b_n = na_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{n+1}{n}\left|\frac{a_{n+1}}{a_n}\right| = 1\cdot\frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} na_n$ converges by the Ratio Test.

29. $\sum_{n=1}^{\infty} n^3 a_n$

SOLUTION Let $b_n = n^3 a_n$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \left|\frac{a_{n+1}}{a_n}\right| = 1^3 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} n^3 a_n$ converges by the Ratio Test.

30. $\sum_{n=1}^{\infty} 2^n a_n$

SOLUTION Let $b_n = 2^n a_n$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot \frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} 2^n a_n$ converges by the Ratio Test.

31. $\sum_{n=1}^{\infty} 3^n a_n$

SOLUTION Let $b_n = 3^n a_n$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \left|\frac{a_{n+1}}{a_n}\right| = 3 \cdot \frac{1}{3} = 1.$$

Therefore, the Ratio Test is inconclusive for the series $\sum_{n=1}^{\infty} 3^n a_n$.

32. $\sum_{n=1}^{\infty} 4^n a_n$

SOLUTION Let $b_n = 4^n a_n$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \left|\frac{a_{n+1}}{a_n}\right| = 4 \cdot \frac{1}{3} = \frac{4}{3} > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} 4^n a_n$ diverges by the Ratio Test.

33. $\sum_{n=1}^{\infty} a_n^2$

SOLUTION Let $b_n = a_n^2$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n^2$ converges by the Ratio Test.

34. Assume that $|a_{n+1}/a_n|$ converges to $\rho = 4$. Does $\sum_{n=1}^{\infty} a_n^{-1}$ converge (assume that $a_n \neq 0$ for all n)?

SOLUTION Let $b_n = a_n^{-1}$. Then

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{4} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n^{-1}$ converges by the Ratio Test.

35. Is the Ratio Test conclusive for the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$?

SOLUTION With $a_n = \frac{1}{n^p}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \left(\frac{n}{n+1}\right)^p \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1^p = 1.$$

Therefore, the Ratio Test is inconclusive for the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

In Exercises 36–41, use the Root Test to determine convergence or divergence (or state that the test is inconclusive).

36. $\sum_{n=0}^{\infty} \frac{1}{10^n}$

SOLUTION With $a_n = \frac{1}{10^n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{10^n}} = \frac{1}{10} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{10} < 1.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{1}{10^n}$ converges by the Root Test.

37. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Root Test.

38. $\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$

SOLUTION With $a_k = \left(\frac{k}{k+10}\right)^k$,

$$\sqrt[k]{a_k} = \sqrt[k]{\left(\frac{k}{k+10}\right)^k} = \frac{k}{k+10} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{a_k} = 1.$$

Therefore, the Root Test is inconclusive for the series $\sum_{k=0}^{\infty} \left(\frac{k}{k+10}\right)^k$. Because

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1 + \frac{10}{k}\right)^{-k} = \lim_{k\to\infty} \left[\left(1 + \frac{10}{k}\right)^{k/10}\right]^{-10} = e^{-10} \neq 0,$$

this series diverges by the Divergence Test.

39. $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

SOLUTION With $a_k = \left(\frac{k}{3k+1}\right)^k$,

$$\sqrt[k]{a_k} = \sqrt[k]{\left(\frac{k}{3k+1}\right)^k} = \frac{k}{3k+1} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{a_k} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$ converges by the Root Test.

40. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$

SOLUTION With $a_k = (1+\frac{1}{n})^{-n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n}} = \left(1+\frac{1}{n}\right)^{-1} \quad \text{and} \quad \lim_{n\to\infty}\sqrt[n]{a_n} = 1^{-1} = 1.$$

Therefore, the Root Test is inconclusive for the series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$.

Because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} = e^{-1} \neq 0,$$

this series diverges by the Divergence Test.

41. $\displaystyle\sum_{n=4}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

SOLUTION With $a_k = (1+\frac{1}{n})^{-n^2}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}} = \left(1+\frac{1}{n}\right)^{-n} \quad \text{and} \quad \lim_{n\to\infty}\sqrt[n]{a_n} = e^{-1} < 1.$$

Therefore, the series $\displaystyle\sum_{n=4}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ converges by the Root Test.

42. Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ diverges. *Hint:* Use $2^{n^2} = (2^n)^n$ and $n! \le n^n$.

SOLUTION Because $n! \le n^n$,

$$\frac{2^{n^2}}{n!} \ge \frac{2^{n^2}}{n^n}.$$

Now, let $a_n = \frac{2^{n^2}}{n^n}$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2^{n^2}}{n^n}} = \frac{2^n}{n},$$

and

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{2^n}{n} = \lim_{x\to\infty}\frac{2^x}{x} = \lim_{x\to\infty}\frac{2^x\ln 2}{1} = \infty > 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n^n}$ diverges by the Root Test. By the Comparison Test, we can then conclude that the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ also diverges.

In Exercises 43–56, determine convergence or divergence using any method covered in the text so far.

43. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+4^n}{7^n}$

SOLUTION Because the series

$$\sum_{n=1}^{\infty}\frac{2^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{4^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{4}{7}\right)^n$$

are both convergent geometric series, it follows that

$$\sum_{n=1}^{\infty}\frac{2^n+4^n}{7^n} = \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n + \sum_{n=1}^{\infty}\left(\frac{4}{7}\right)^n$$

also converges.

44. $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

SOLUTION The presence of the factorial suggests applying the Ratio Test. With $a_n = \frac{n^3}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} = \frac{(n+1)^2}{n^3} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{n!}$ converges by the Ratio Test.

45. $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

SOLUTION The presence of the exponential term suggests applying the Ratio Test. With $a_n = \frac{n^3}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{1}{5}\left(1+\frac{1}{n}\right)^3 \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} \cdot 1^3 = \frac{1}{5} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by the Ratio Test.

46. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

SOLUTION The general term in this series suggests applying the Integral Test. Let $f(x) = \frac{1}{x(\ln x)^3}$. This function is continuous, positive and decreasing for $x \ge 2$, so the Integral Test does apply. Now

$$\int_2^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{R\to\infty}\int_2^R \frac{dx}{x(\ln x)^3} = \lim_{R\to\infty}\int_{\ln 2}^{\ln R} \frac{du}{u^3} = -\frac{1}{2}\lim_{R\to\infty}\left(\frac{1}{(\ln R)^2} - \frac{1}{(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}.$$

The integral converges; hence, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ also converges.

47. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-n^2}}$

SOLUTION This series is similar to a p-series; because

$$\frac{1}{\sqrt{n^3-n^2}} \approx \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

for large n, we will apply the Limit Comparison Test comparing with the p-series with $p = \frac{3}{2}$. Now,

$$L = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^3-n^2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\sqrt{\frac{n^3}{n^3-n^2}} = 1.$$

The p-series with $p = \frac{3}{2}$ converges and L exists; therefore, the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-n^2}}$ also converges.

48. $\sum_{n=1}^{\infty} \frac{n^2+4n}{3n^4+9}$

SOLUTION This series is similar to a p-series; because

$$\frac{n^2+4n}{3n^4+9} \approx \frac{n^2}{\sqrt{3n^4}} = \frac{1}{3n^2}$$

for large n, we will apply the Limit Comparison Test comparing with the p-series with $p = 2$. Now,

$$L = \lim_{n\to\infty}\frac{\frac{n^2+4n}{3n^4+9}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^4+4n^3}{3n^4+9} = \frac{1}{3}.$$

The p-series with $p = 2$ converges and L exists; therefore, the series $\sum_{n=1}^{\infty} \frac{n^2+4n}{3n^4+9}$ also converges.

49. $\displaystyle\sum_{n=1}^{\infty} n^{-0.8}$

SOLUTION

$$\sum_{n=1}^{\infty} n^{-0.8} = \sum_{n=1}^{\infty} \frac{1}{n^{0.8}}$$

so that this is a divergent p-series.

50. $\displaystyle\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$

SOLUTION

$$\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8} = \sum_{n=1}^{\infty} (0.8^{-1})^n n^{-0.8} = \sum_{n=1}^{\infty} \frac{1.25^n}{n^{0.8}}$$

With $a_n = \frac{1.25^n}{n^{0.8}}$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1.25^{n+1}}{(n+1)^{0.8}} \cdot \frac{n^{0.8}}{1.25^n} = 1.25\left(\frac{n}{n+1}\right)^{0.8}$$

so that

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1.25 > 1$$

Thus the original series diverges, by the Ratio Test.

51. $\displaystyle\sum_{n=1}^{\infty} 4^{-2n+1}$

SOLUTION Observe

$$\sum_{n=1}^{\infty} 4^{-2n+1} = \sum_{n=1}^{\infty} 4 \cdot (4^{-2})^n = \sum_{n=1}^{\infty} 4\left(\frac{1}{16}\right)^n$$

is a geometric series with $r = \frac{1}{16}$; therefore, this series converges.

52. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

SOLUTION This is an alternating series with $a_n = \frac{1}{\sqrt{n}}$. Because a_n forms a decreasing sequence which converges to zero, tl series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Leibniz Test.

53. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

SOLUTION Here, we will apply the Limit Comparison Test, comparing with the p-series with $p = 2$. Now,

$$L = \lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{u\to 0} \frac{\sin u}{u} = 1,$$

where $u = \frac{1}{n^2}$. The p-series with $p = 2$ converges and L exists; therefore, the series $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ also converges.

54. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\frac{1}{n}$

SOLUTION Because

$$\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1 \neq 0,$$

the general term in the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\frac{1}{n}$ does not tend toward zero; therefore, the series diverges by the Divergence Test.

55. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

SOLUTION Because

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n}} = \lim_{x\to\infty} \frac{2^x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2^x \ln 2}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2^{x+1}\sqrt{x}\ln 2 = \infty \neq 0,$$

the general term in the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$ does not tend toward zero; therefore, the series diverges by the Divergence Test.

56. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

SOLUTION Because the general term has the form of a function of n raised to the nth power, we might be tempted to use the Root Test; however, the Root Test is inconclusive for this series. Instead, note

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{12}{n}\right)^{-n} = \lim_{n\to\infty} \left[\left(1 + \frac{12}{n}\right)^{n/12}\right]^{-12} = e^{-12} \neq 0.$$

Therefore, the series diverges by the Divergence Test.

Further Insights and Challenges

57. **Proof of the Root Test** Let $S = \sum_{n=0}^{\infty} a_n$ be a positive series, and assume that $L = \lim_{n \to \infty} \sqrt[n]{a_n}$ exists.

(a) Show that S converges if $L < 1$. *Hint:* Choose R with $L < R < 1$ and show that $a_n \leq R^n$ for n sufficiently large. Then compare with the geometric series $\sum R^n$.

(b) Show that S diverges if $L > 1$.

SOLUTION Suppose $\lim_{n \to \infty} \sqrt[n]{a_n} = L$ exists.

(a) If $L < 1$, let $\epsilon = \frac{1-L}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \leq \sqrt[n]{a_n} - L \leq \epsilon$$

for $n \geq N$. From this, we conclude that

$$0 \leq \sqrt[n]{a_n} \leq L + \epsilon$$

for $n \geq N$. Now, let $R = L + \epsilon$. Then

$$R = L + \frac{1-L}{2} = \frac{L+1}{2} < \frac{1+1}{2} = 1,$$

and

$$0 \leq \sqrt[n]{a_n} \leq R \quad \text{or} \quad 0 \leq a_n \leq R^n$$

for $n \geq N$. Because $0 \leq R < 1$, the series $\sum_{n=N}^{\infty} R^n$ is a convergent geometric series, so the series $\sum_{n=N}^{\infty} a_n$ converges by the Comparison Test. Therefore, the series $\sum_{n=0}^{\infty} a_n$ also converges.

(b) If $L > 1$, let $\epsilon = \frac{L-1}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \leq \sqrt[n]{a_n} - L \leq \epsilon$$

for $n \geq N$. From this, we conclude that

$$L - \epsilon \leq \sqrt[n]{a_n}$$

for $n \geq N$. Now, let $R = L - \epsilon$. Then

$$R = L - \frac{L-1}{2} = \frac{L+1}{2} > \frac{1+1}{2} = 1,$$

and

$$R \leq \sqrt[n]{a_n} \quad \text{or} \quad R^n \leq a_n$$

for $n \geq N$. Because $R > 1$, the series $\sum_{n=N}^{\infty} R^n$ is a divergent geometric series, so the series $\sum_{n=N}^{\infty} a_n$ diverges by the Comparison Test. Therefore, the series $\sum_{n=0}^{\infty} a_n$ also diverges.

58. Show that the Ratio Test does not apply, but verify convergence using the Comparison Test for the series

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots$$

SOLUTION The general term of the series is:

$$a_n = \begin{cases} \dfrac{1}{2^n} & n \text{ odd} \\ \dfrac{1}{3^n} & n \text{ even} \end{cases}$$

First use the Ratio Test. If n is even,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \frac{3^n}{2^{n+1}} = \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n$$

whereas, if n is odd,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{3^{n+1}}}{\frac{1}{2^n}} = \frac{2^n}{3^{n+1}} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$$

Since $\lim_{n\to\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n = 0$ and $\lim_{n\to\infty} \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n = \infty$, the sequence $\frac{a_{n+1}}{a_n}$ does not converge, and the Ratio Test is inconclusive.

However, we have $0 \leq a_n \leq \frac{1}{2^n}$ for all n, so the series converges by comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

59. Let $S = \sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$, where c is a constant.

(a) Prove that S converges absolutely if $|c| < e$ and diverges if $|c| > e$.

(b) It is known that $\lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}$. Verify this numerically.

(c) Use the Limit Comparison Test to prove that S diverges for $c = e$.

SOLUTION

(a) With $a_n = \frac{c^n n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|c|^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{|c|^n n!} = |c| \left(\frac{n}{n+1}\right)^n = |c| \left(1 + \frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |c| e^{-1}.$$

Thus, by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$ converges when $|c|e^{-1} < 1$, or when $|c| < e$. The series diverges when $|c| > e$.

(b) The table below lists the value of $\frac{e^n n!}{n^{n+1/2}}$ for several increasing values of n. Since $\sqrt{2\pi} = 2.506628275$, the numerical evidence verifies that

$$\lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}.$$

n	100	1000	10000	100000
$\frac{e^n n!}{n^{n+1/2}}$	2.508717995	2.506837169	2.506649163	2.506630363

(c) With $c = e$, the series S becomes $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$. Using the result from part (b),

$$L = \lim_{n\to\infty} \frac{\frac{e^n n!}{n^n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}.$$

Because the series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by the Divergence Test and $L > 0$, we conclude that $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ diverges by the Limit Comparison Test.

10.6 Power Series

Preliminary Questions

1. Suppose that $\sum a_n x^n$ converges for $x = 5$. Must it also converge for $x = 4$? What about $x = -3$?

SOLUTION The power series $\sum a_n x^n$ is centered at $x = 0$. Because the series converges for $x = 5$, the radius of convergence must be at least 5 and the series converges absolutely at least for the interval $|x| < 5$. Both $x = 4$ and $x = -3$ are inside this interval, so the series converges for $x = 4$ and for $x = -3$.

2. Suppose that $\sum a_n (x-6)^n$ converges for $x = 10$. At which of the points (a)–(d) must it also converge?

(a) $x = 8$ **(b)** $x = 11$ **(c)** $x = 3$ **(d)** $x = 0$

SOLUTION The given power series is centered at $x = 6$. Because the series converges for $x = 10$, the radius of convergence must be at least $|10 - 6| = 4$ and the series converges absolutely at least for the interval $|x - 6| < 4$, or $2 < x < 10$.

(a) $x = 8$ is inside the interval $2 < x < 10$, so the series converges for $x = 8$.

(b) $x = 11$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 11$.

(c) $x = 3$ is inside the interval $2 < x < 10$, so the series converges for $x = 2$.

(d) $x = 0$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 0$.

3. What is the radius of convergence of $F(3x)$ if $F(x)$ is a power series with radius of convergence $R = 12$?

SOLUTION If the power series $F(x)$ has radius of convergence $R = 12$, then the power series $F(3x)$ has radius of convergence $R = \frac{12}{3} = 4$.

4. The power series $F(x) = \sum_{n=1}^{\infty} n x^n$ has radius of convergence $R = 1$. What is the power series expansion of $F'(x)$ and what is its radius of convergence?

SOLUTION We obtain the power series expansion for $F'(x)$ by differentiating the power series expansion for $F(x)$ term-by-term. Thus,

$$F'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}.$$

The radius of convergence for this series is $R = 1$, the same as the radius of convergence for the series expansion for $F(x)$.

Exercises

1. Use the Ratio Test to determine the radius of convergence R of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. Does it converge at the endpoints $x = \pm R$?

SOLUTION With $a_n = \frac{x^n}{2^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{|x|^n} = \frac{|x|}{2} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2}.$$

By the Ratio Test, the series converges when $\rho = \frac{|x|}{2} < 1$, or $|x| < 2$, and diverges when $\rho = \frac{|x|}{2} > 1$, or $|x| > 2$. The radius of convergence is therefore $R = 2$. For $x = -2$, the left endpoint, the series becomes $\sum_{n=0}^{\infty} (-1)^n$, which is divergent. For $x = 2$, the right endpoint, the series becomes $\sum_{n=0}^{\infty} 1$, which is also divergent. Thus the series diverges at both endpoints.