

Sequences and Series

What's the difference...
between a sequence and a
series?

A sequence is a list (separated by commas).

A series adds the numbers in the list together.

Example:

Sequence: 1, 2, 3, 4, ..., n, ...

Series: $1 + 2 + 3 + 4 + \dots + n + \dots$

(note that in calculus we only examine infinite sequences
and series)

What symbol(s) do we use

For a sequence?

For a series?

$\{a_n\}$ represents a sequence

$\sum a_n$ represents a series

How do you find the limit of a sequence?

$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \dots, a_{100}, a_{101}, \dots, a_{1000},$
 $a_{1001}, a_{1002}, \dots, a_{7000000}, a_{7000001}, \dots, a_n, \dots$

→ where's it going?

Just take the limit as $n \rightarrow \infty$

Remember, you can treat "n" as if it were an "x"

(You may have to use L'Hopital's Rule)

There are 2 special series that we can actually find the sum of...

Geometric and Telescoping

What does a geometric series look like?

How do you find it's sum?

Why is it called geometric?

Geometric series are of the form:

$$\sum a(r)^n$$

A geometric series only converges if r is between -1 and 1

The sum of a convergent geometric series is:

$$\frac{\textit{the} \cdot \textit{first} \cdot \textit{term}}{1 - r}$$

See the next slide for a possible answer as to why these series are called "geometric"

Maybe this is why the name “geometric” since the idea originated from a “physical” problem...

The ancient Greek philosopher **Zeno** (5th century BC) was famous for creating paradoxes to vex the intellectuals of his time. In one of those paradoxes, he says that if you are 1 meter away from a wall, you can never reach the wall by walking toward it. This is because first you have to traverse half the **distance**, or $1/2$ meter, then half the remaining distance, or $1/4$ meter, then half again, or $1/8$ meter, and so on. You can never reach the wall because there is always some small finite distance left. The theory of infinite geometric series can be used to answer this paradox. Zeno is actually saying that we cannot get to the wall because the total distance we must travel is $1/2 + 1/4 + 1/8 + 1/16 + \dots$, an infinite sum. But this is just an infinite geometric series with first term $1/2$ and common ratio $1/2$, and its sum is $(1/2)/(1 - 1/2) = 1$. So the infinite sum is one meter and we can indeed get to the wall.

What about telescoping (or “collapsing”) series?

What are telescoping series?

What types of series do you suspect of being telescoping and how do you find their sum?



If when expanded, all the terms “in the middle” cancel out and you are left with only the first term(s) because the nth term heads to zero, then the series is “telescoping” or “collapsing”

Suspicious forms:

$$\sum \left(\frac{1}{an+b} \pm \frac{1}{cn+d} \right) \quad \text{or} \quad \sum \frac{1}{(an+b)(cn+d)}$$

The latter can be separated into 2 fractions and then observed.
Always write out the first few terms as well as the last nth terms in order to observe the cancelling pattern.

Also! Make sure that the non cancelling nth term goes to zero.

Telescoping series can be cleverly disguised!

So be on the look out for them.

In general, to find S , the sum of a series, you need to take the limit of the partial sums: S_n

What's a partial Sum?

In general, to find S , the sum of a series, you need to take the limit of the partial sums: S_n

What's a partial Sum?

...

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

In other words:

$$\sum a_n = \lim_{n \rightarrow \infty} S_n = S \text{ (If } S \text{ exists)}$$

$$\text{If } a_n \not\rightarrow 0$$

What does that tell you about the series?

...

The series diverges.

What if $a_n \rightarrow 0$?

...

Then the series ***might*** converge.

Alternating Series Test

What does it say?

If the terms of a series alternate positive and negative
AND also go to zero, the series will converge.

Often there will be $(-1)^n$ in the formula...but check it out and make sure the terms reeeeeally alternate. Don't be tricked!

Also note that if the series alternates,
and if you stop adding at a_n ,
your error will be less than the next term: a_{n+1}

OK...here's a couple of famous series that come in handy quite often.

What are p-series
and

What is the harmonic series?

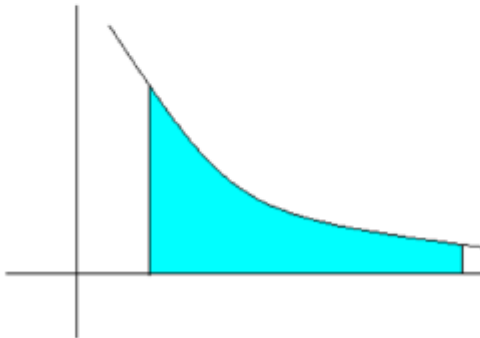
The harmonic series:
diverges – most
people are surprised!

$$\sum \frac{1}{n}$$

p-series:
converges for $p > 1$
diverges for $p < 1$

$$\sum \frac{1}{n^p}$$

What's the integral test
and
When should you use it?



The integral test says that if

$$\int_c^{\infty} f(x)dx = K$$

where K is a positive real number,
then the series converges also.
...but NOT to the same number!

(you can however use $\int_n^{\infty} f(x)dx$ to approximate the error for S_n if n is large)

If the integral diverges, then so does the series.

Use the integral test only if changing n to x yields an easily integratable function.

Now we're moving along!!

Here are three limits you need to know...

as $n \rightarrow \infty$

what happens to:

1. $\sqrt[n]{c}$

2. $\sqrt[n]{n}$

and finally

3. $\left(1 + \frac{c}{n}\right)^n$

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...

The answers are 1, 1, and e^c respectively.

Next question:

What is the Root Test and when should you use it?

The root test says: that as $n \rightarrow \infty$

If $\sqrt[n]{|a_n|} \rightarrow b < 1$ the series converges.

If $\sqrt[n]{|a_n|} \rightarrow b > 1$ the series diverges

But if $\sqrt[n]{|a_n|} \rightarrow b = 1$ the test is inconclusive

Use the ROOT TEST

when the terms have n 's in their exponents.

What is the **RATIO TEST**?

When should you use it?

The **RATIO TEST** should be used

when a_n contains $n!$

or something like $n!$

such as:

$$1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n + 1)$$

It says to compare the limit as $n \rightarrow \infty$ of $\frac{a_{n+1}}{a_n}$ to 1

A limit < 1 indicates convergence, > 1 indicates divergence

If the limit equals 1 then the test is inconclusive.

OK...just 2 more tests for convergence...

Comparison Tests:

Direct Comparison

&

Limit Comparison

Direct Comparison...

What is it?

When do you use it?

If you can show that your positive terms are greater than a known divergent series

$$\text{(like } \sum \frac{1}{n} \text{ or a p-series where } p < 1)$$

or smaller than a known convergent series

$$\text{(like a p-series where } p > 1)$$

then you are using the Direct Comparison Test.

...

Question: If it is not easy to compare the series directly, how do you employ the Limit Comparison Test??

Form a ratio with the terms of the series you are testing for convergence and the terms of a

$$\text{known series that is similar: } \frac{a_n}{b_n}$$

If the limit of this ratio as $n \rightarrow \infty$ is a positive real #

then both series “do the same thing”

i.e. both converge or both diverge

If the limit is zero or infinity...then either you are comparing your series to one that is not similar enough...or...you need a different test.

What is a Power Series?

A power series is of the form:

$$\sum a_n x^n$$

So...how do you figure out
the values of x which yield
convergence?

Put absolute value around the x part and apply either the ratio or the root test.

For example: $\sum \frac{1}{n}(x-2)^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}|x-2|^n} = \lim_{n \rightarrow \infty} \frac{|x-2|}{\sqrt[n]{n}} = |x-2| < 1$$

Now solve for x: $-1 < x-2 < 1$
 $1 < x < 3$

Checking the endpoints separately, $x=3$ yields the harmonic series (divergent) and $x=1$ yields the alternating harmonic series (convergent).

Interval of convergence is $[1, 3)$, radius of convergence = 1

What is the Binomial Series Formula?

$$(1+x)^s = \sum_{n=0}^{\infty} \frac{s(s-1)(s-2)\dots(s-(n-1))}{n!} x^n$$

Remember that the fraction has the same number of fractions (or integers if s is an integer) in the numerator as the factorial in the denominator.

Also...the interval of convergence is (-1,1)

Example:

$$(1+x^2)^{\frac{2}{3}} = 1 + \frac{\frac{2}{3}}{1}(x^2)^1 + \frac{(\frac{2}{3})(\frac{-1}{3})}{2 \bullet 1}(x^2)^2 + \frac{(\frac{2}{3})(\frac{-1}{3})(\frac{-4}{3})}{3 \bullet 2 \bullet 1}(x^2)^3 + \dots$$

Here come some expressions you should have memorized the infinite series for...

$$\frac{1}{1-x} = ?$$

Where $-1 < x < 1$

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Ready?

$$\frac{1}{1+x} = ?$$

Where $-1 < x < 1$

$$1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

Ready?

$$\ln(1 + x) = ?$$

Never can remember that one...
so I just integrate the previous one.

$$\int \frac{1}{1+x} dx = \int 1 - x + x^2 - x^3 + \dots dx$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$\sin x = ???$

$\cos x = ???$

$\tan^{-1}x = ???$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

Note the similarities...if you know one, do you know the rest?

...

Just four more questions!

What is the formula for a Maclauren Series?

(Used to approximate a function near zero)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

What is the formula
for a Taylor
Series?

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Used to approximate $f(x)$ near a .

What is the LaGrange
Remainder Formula for
approximating errors in
NON alternating series?

Given: $f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n$

$$|R_n| = \left| \frac{f^{(n+1)}(t_x)}{(n+1)!} (x-a)^{n+1} \right|$$

Where t_x is some number between a and x

Then we find the maximum possible value of $f^{(n+1)}(t_x)$

to approximate the error (remainder).

Last question!!!

How do you approximate the error (remainder) for an alternating series?

...

The error in an alternating series
is always less than the next term.

$$|R_n| < |a_{n+1}|$$

Be sure to check out the power point drills for:

Pre-Calc Topics, Derivatives, Integrals,
Miscellaneous Topics, and other BC/BCD Topics