CH.9 INFINITE SERIES

9-1 Sequences

Sequence: A **sequence** is a list of numbers, called **terms**, in a definite order. Sequences of objects are most commonly denoted using braces.

$**$If the limit of a sequence exists, then we say the sequence **converges**. Otherwise the sequence **diverges**.

EX#1: Given $\left\{(-1)^n \frac{3n+5}{n^2-2}\right\}$ find the third term:___________

Recursively defined sequence: Each subsequent term depends on the previous term.

EX#2: $a_{k+1} = 5a_k + 10$; $a_1 = -2$

$a_2 =$ $a_3 =$ $a_4 =$ $a_5 =$

Most famous recursively defined sequence is the **Fibonacci Sequence:** $a_{k+2} = a_k + a_{k+1}$; $a_1 = 1,\ a_2 = 1$

Fibonacci Sequence: 1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , _____ , _____ , _____ , ...

Fibonacci Petals

3 petals ⇒ lily, iris

5 petals ⇒ buttercup, wild rose, larkspur, columbine

8 petals ⇒ delphiniums

13 petals ⇒ ragwort, corn marigold, cineraria

21 petals ⇒ aster, black-eyed susan, chicory

34 petals ⇒ plantain, pytethrum

55, 89 petals ⇒ michelmas daisies, the asteraceae family

Humans exhibit Fibonacci characteristics.

The Golden Ratio is seen in the proportions in the sections of a finger.

EX#3: **Determine the convergence or divergence of each sequence**

$a)\ a_n = \left\{\frac{3n^4+5}{2n^2-7}\right\}$ $b)\ a_n = \left\{\frac{7n-2}{8n^2-4}\right\}$ $c)\ a_n = \left\{\frac{6n+8}{\sqrt{9n^2+1}}\right\}$ $d)\ a_n = \left\{5 - \frac{1}{n}\right\}$

EX#4: **Simplify each**

$a)\ \frac{20!}{19!}$ $b)\ \frac{7!}{10!}$ $c)\ \frac{(n+1)!}{n!}$ $d)\ \frac{(n+1)!}{(n+2)!}$

$e)\ \frac{(n+3)!}{n!}$ $f)\ \frac{6^n}{6^{n+1}}$ $g)\ \frac{(x+2)^{n+1}}{(x+2)^n}$ $h)\ \frac{x^{n+1}\cdot 5^n}{x^n \cdot 5^{n+1}}$

9-2 Series and Convergence

Infinite series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots + a_n + \ldots\ldots$$

Definition of Convergent and Divergent series:

For the infinite series $\sum a_n$, the nth partial sum is given by $S_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots + a_n$.

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum a_n$ converges.

The limit S is called the sum of the series. $S = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots + a_n + \ldots\ldots$

If $\{S_n\}$ diverges, then the series diverges.

Geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^n + \ldots\ldots \quad a \neq 0$$

***Geometric series Test:

A geometric series $\sum_{n=m}^{\infty} a \cdot r^n$ converges iff $|r| < 1$. A geometric series $\sum_{n=m}^{\infty} a \cdot r^n$ diverges iff $|r| \geq 1$.

If a geometric series converges, it converges to the **Sum:** $S = \dfrac{a_1}{1-r}$

*** **nth term test** (Used to show immediate divergence)

If $\lim_{n\to\infty} a_n = 0$ then it may converge. (bottom power is greater and we must proceed and use a different test, because this test cannot prove convergence).

If $\lim_{n\to\infty} a_n \neq 0$ then it diverges (top power is the same or greater than the bottom power).

EX #1: Test for Convergence

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$ b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ c) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1}$

EX #2: A ball is dropped from a height of 6 feet and begins bouncing. The height of each bounce is $\frac{3}{4}$ the height of the previous bounce. Find the total vertical distance travelled by the ball.

*****Telescoping Series are convergent** (Limit = 0 and the terms get smaller as we approach ∞)

Find the sum of the following telescoping series.

EX #3: $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

EX #4: $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$

EX #5: $\sum_{n=2}^{\infty} \frac{4}{n(n+1)}$

Repeating decimals

0.222222....

0.333333....

0.242424....

0.833333....

9-3 Integral Test / p-series

***Integral Test:** If f is positive, continuous and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx \text{ either both converge or both diverge.}$$

EX #1: Use Integral Test to test for convergence

$a)\ \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$b)\ \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

***p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1) Converges if $p > 1$
2) Diverges if $0 < p \leq 1$

If $p = 1$, it is called the harmonic series. $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series

EX #2: Use p-series to test for convergence

$a)\ \sum_{n=1}^{\infty} \frac{1}{n^2}$

$b)\ \sum_{n=1}^{\infty} \frac{3}{n}$

$c)\ \sum_{n=1}^{\infty} \frac{7}{n^{3/2}}$

EX #3: Review (Test each series for convergence)

$a)\ \sum_{n=1}^{\infty} 7\left(\frac{9}{10}\right)^n$

$b)\ \sum_{n=1}^{\infty} \frac{n^2-7}{n+5}$

$c)\ \sum_{n=2}^{\infty} \frac{6}{n-1} - \frac{6}{n+1}$

$d)\ \sum_{n=1}^{\infty} \left(\frac{11}{7}\right)^n$

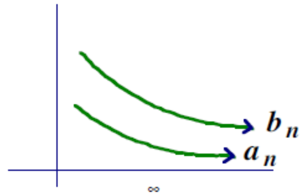
9-4 Comparison of Series

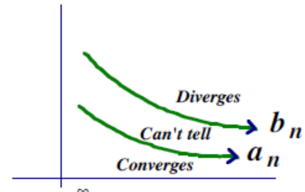
*****Direct Comparison Test**

Let $0 \le a_n \le b_n$ for all n.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.





If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges.

*****Limit Comparison Test**

Suppose then $a_n > 0$, $b_n > 0$, and $\lim_{x\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Assume $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

*** Limit test works well when comparing "messy" algebraic series with a p-series. In choosing an appropriate p-series, you must choose one with an n^{th} term of the same magnitude as the n^{th} term of the given series.

Determine Convergence or Divergence for each

EX #1: $\sum_{n=1}^{\infty} \frac{1}{n-8}$

EX #2: $\sum_{n=1}^{\infty} \frac{1}{n^2+14}$

EX #3: $\sum_{n=1}^{\infty} \frac{2^n}{5+3^n}$

EX #4: $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

EX #5: $\sum_{n=1}^{\infty} \frac{n}{2n^2+5}$

EX #6: $\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^2+1}$

9-5 Alternating Series

*****Alternating Series Test**

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ **converge**
if the following **two conditions are met.**

1) $\lim_{n\to\infty} a_n = 0$ (the bottom is bigger than the top)
2) $a_{n+1} \le a_n$ for all n. (each succeeding term is getting smaller than the preceding term)

*Alternating Series are more likely to be convergent than other series.
***This test does not prove divergence.**
*If the condtions are not met, usually bottom not smaller than the top, then the series diverges by nth term test.

EX #1: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

EX #2: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$

***Alternating Series Remainder**

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of remainder R_n involved in approximating the sum S by S_n **is less than or equal to the first neglected term**.
That is,
$|S - S_n| = |R_n| \le a_{n+1}$

EX #3: Approximate the sum of the series by its first 6 terms (S_6)
$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$

EX#4: Find the remainder (error).

Absolute Convergence (Alternating Series)

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Definition ofAbsolute and Conditional Convergence

1) $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
2) $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Determine if each series is for absolutely convergent, conditionally convergent, or divergent

EX #1: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

EX #2: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$

EX #3: $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3n+1}$

EX #4: $\sum_{n=1}^{\infty} (-1)^n \left(\frac{8}{9}\right)^n$

EX #5: $\sum_{n=1}^{\infty} (-1)^n 2$

EX #6: $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

9-6 The Ratio and Root Test

***__Ratio Test:__** Let $\sum a_n$ be a series with nonzero terms

1) $\sum a_n$ <u>converges absolutely</u> if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

2) $\sum a_n$ <u>diverges</u> if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

3) The ratio test is <u>inconclusive</u> if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

EX: $\sum_{n=1}^{\infty}\frac{(-1)^n n^5}{2^n}$ $\quad \lim_{n\to\infty}\left|\frac{\frac{(n+1)^5}{2^{n+1}}}{\frac{n^5}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^5}{2^{n+1}}\cdot\frac{2^n}{n^5}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^5}{2n^5}\right| = \frac{1}{2}$ so series Converges

***Root Test:** Let $\sum a_n$ be a series with nonzero terms

1) $\sum a_n$ converges absolutely if $\lim_{x\to\infty} \sqrt[n]{|a_n|} < 1$

2) $\sum a_n$ diverges if $\lim_{x\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$

3) The root test is inconclusive if $\lim_{x\to\infty} \sqrt[n]{|a_n|} = 1$

EX #1: $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!}$

EX #2: $\sum_{n=0}^{\infty} \frac{n^2 \cdot 2^{n+1}}{3^n}$

EX#3: $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

EX #4: $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Summary of tests for Series

Test	Series	Converges	Diverges	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	if $\int_1^{\infty} f(x)dx$ converges $\int_1^{\infty} f(x)dx$ is finite	if $\int_1^{\infty} f(x)dx$ diverges $\int_1^{\infty} f(x)dx = \infty$	Remainder: $0 < R_N < \int_N^{\infty} f(x)dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$.
Direct comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	Either show your series is less than a convergent series or greater than a divergent series.
Limit Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	If you know your series is convergent then compare your series to a convergent series and vice versa.

9-7 Taylor Polynomials and Approximations

Definition of nth Taylor Polynomial and MacLaurin Polynomial ($c = 0$)

If f has n derivatives at c, then the polynomial:

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots\ldots + \frac{f^n(c)}{n!}(x-c)^n$$

is the **nth Taylor Polynomial** for f at c.

If $c = 0$, then

$$P_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \ldots\ldots + \frac{f^n(0)}{n!}(x-0)^n$$

is called the **nth MacLaurin polynomial** for f.

Taylor series of $f(x)$ $\rightarrow$ this is a power series $\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$

EX #1: Find the Taylor Polynomials $P_4(x)$ for $f(x) = \ln x$ centered at 1.

EX #2: Find the fifth degree MacLaurin Series $(\text{centered at } c = 0)$ for $f(x) = \sin x$.

Find the fourth degree MacLaurin Series for each.

EX #3: $f(x) = \cos x$.

EX #4: $f(x) = e^x$

EX #5: Find the fourth degree Taylor Polynomial for $f(x) = \sin x$ centered at $c = \frac{\pi}{4}$.

<u>**Formulas for sign changes**</u>

$$(-1)^{n(n+1)/2} = + - - + + - - + +$$

$$(-1)^{(n+1)(n+2)/2} = - - + + - - + +$$

$$(-1)^{(n+2)(n+3)/2} = - + + - - + + - -$$

$$(-1)^{(n+3)(n+4)/2} = + + - - + + - -$$

9-8 Power Series

When finding the interval of convergence we use either Geometric Series Test, Root Test or Ratio Test.

Find the interval of convergence, radius of convergence and the center for each example below.

EX #1: $\sum_{n=0}^{\infty} 3(x-2)^n$

interval of convergence:

radius of convergence:

center:

EX #2: $\sum_{n=1}^{\infty} \frac{x^n}{n}$

interval of convergence:

radius of convergence:

center:

EX #3: $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

interval of convergence:

radius of convergence:

center:

EX #4: $\sum_{n=0}^{\infty} \frac{(x-8)^n}{n \cdot 15^n}$

interval of convergence:

radius of convergence:

center:

Special Cases

When finding the interval of convergence :

If $\lim_{n\to\infty}\left|\ \right| = 0$ then the power series converges from $(-\infty, \infty)$.

If $\lim_{n\to\infty}\left|\ \right| = \infty$ then the power series converges at the center only.

EX #5 : $\sum_{n=0}^{\infty} \frac{n!(x+9)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x+9)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!(x+9)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)(x+9)}{3}\right| = \infty$$

Never less than 1.

Converges at center -9 only.

interval of convergence : Converges at center -9 only

radius of convergence : 0

center : -9

EX #6 : $\sum_{n=0}^{\infty} \frac{(x-7)^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{(x-7)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-7)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-7)}{n+1}\right| = 0$$

Always less than 1.

Converges always. $(-\infty, \infty)$

interval of convergence : $-\infty < x < \infty$

radius of convergence : ∞

center : 7

9-9 Representation of Functions by Power Series

$\sum_{n=0}^{\infty} a_1 r^n = \frac{a_1}{1-r}, |r| < 1$ We will work backward and convert $\frac{a_1}{1-r}$ to $\sum_{n=0}^{\infty} a_1 r^n$.

There are two techniques to convert $\frac{a_1}{1-r}$ to a power series.

1) Rewrite function to look like $\frac{a_1}{1-r}$.

2) Use Long Division

Write the power series and find the interval of convergence for each example below.

EX #1: $f(x) = \frac{1}{2-x}, \quad c = 0$

EX #2: $f(x) = \frac{1}{2-x}, \quad c = 5$

EX #3: $f(x) = \frac{1}{2x-5}, \quad c = 0$

EX #4: $f(x) = \frac{1}{2x-5}, \quad c = 3$

Power Series for Elementary Functions

Function		Interval of Convergence
$\frac{1}{x}$	$= 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x}$	$= 1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$	$-1<x<1$
$\frac{1}{1-x}$	$= 1+x+x^2+x^3+x^4+x^5+\cdots+x^n+\cdots$	$-1<x<1$
$\ln x$	$= (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\le 2$
e^x	$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$

Power Series for Elementary Functions (... continued)

Function	**Interval of Convergence**
$\sin x \; = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dfrac{x^9}{9!}-\cdots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x \; =1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dfrac{x^8}{8!}-\cdots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\dfrac{x^9}{9}-\cdots+\dfrac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x = x+\dfrac{x^3}{2\cdot 3}+\dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\dfrac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$
$(1+x)^k=1+kx+\dfrac{k(k-1)x^2}{2!}+\dfrac{k(k-1)(k-2)x^3}{3!}+\dfrac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$	$-1<x<1$*

*The convergence at $x=\pm 1$ depends on the value of k.

9-10 Taylor and MacLaurin Series

We will find other series by adjusting known series. We can adjust by adding/subtracting, multiplying/dividing, replacing(substituting) or taking a derivative/integral.

Taylor Series to memorize :

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \cdots$$

EX #1 : Given $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad f(x) = x^4 - \frac{x^7}{6!} + \frac{x^{10}}{11!} - \frac{x^{13}}{16!}$

Find each:

We can multiply:

$x\cos x =$ $\qquad$ $x^4 f(x) =$

We can substitute:

$\cos 2x =$ $\qquad$ $f(x^3) =$

We can differentiate/integrate:

$\sin x =$ $\qquad$ $f'(x) =$

We can combo:

$\sin 3x =$ $\qquad$ $x \cdot f(4x) =$

EX #2 : Given $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ $\qquad$ Find $\arctan x$ $\left(\text{Hint: } \frac{d}{dx}\arctan x = \frac{1}{1+x^2}\right)$

EX #3: Given $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ Find $\int_0^1 e^{-x^2}\,dx$

EX #4: Given $g(x) = 1 + \frac{x^4}{6!} + \frac{x^8}{12!} + \frac{x^{12}}{18!} + \cdots$ Find $x^2 \cdot g(x^4)$

EX: Find $\sin^2 x$ $\quad (\sin^2 x = \frac{1-\cos 2x}{2})$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots$$

$$1 - \cos 2x = \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots$$

$$\frac{1-\cos 2x}{2} = \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} - \ldots \quad = \frac{4x^2}{2\cdot 2!} - \frac{16x^4}{2\cdot 4!} + \frac{32x^6}{2\cdot 6!} - \ldots \quad = x^2 - \frac{x^4}{3} + \frac{x^6}{45} - \ldots$$

Lagrange Error Bound

$Error = \left|f(x)-P_n(x)\right| \le \left|R_n(x)\right| \quad where \;\; R_n(x) \le \dfrac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}$

$f^{n+1}(z)$ is the <u>MAXIMUM</u> of $(n+1)^{th}$ derivative of the function

EX: Given the 4th degree Maclaurin Series for $\cos x$.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}.....$$

Use Maclaurin Series to approximate value of $\cos(0.1)$.

$P_4(.1) \doteq 1-\dfrac{(0.1)^2}{2!}+\dfrac{(0.1)^4}{4!} \doteq 0.9950041667$ $\quad$ (Actual value of $\cos(0.1)=0.9950041653$)

EX: Lagrange Error Bound at 0.1

$f^5(x) = -\sin x$ $\quad$ (The MAXIMUM of $-\sin x$ is 1)

$\left|\cos x - P_4(x)\right| \le \dfrac{(1)(0.1-0)^5}{5!} = 0.000000083$ $\quad$ (Actual gap: 0.0000000014)

EX #1: $f(1)=2, \quad f'(1)=5, \quad f''(1)=7, \quad f'''(1)=12$

a) Write a 2nd Degree Taylor Polynomial

b) Use Taylor Polynomial to approximate 1.1

c) Lagrange Error Bound at 1.1

87. The function f has derivatives of all orders for all real numbers, and $f^{(4)}(x) = e^{\sin x}$. If the third-degree Taylor polynomial for f about $x = 0$ is used to approximate f on the interval $[0, 1]$, what is the Lagrange error bound for the maximum error on the interval $[0, 1]$?

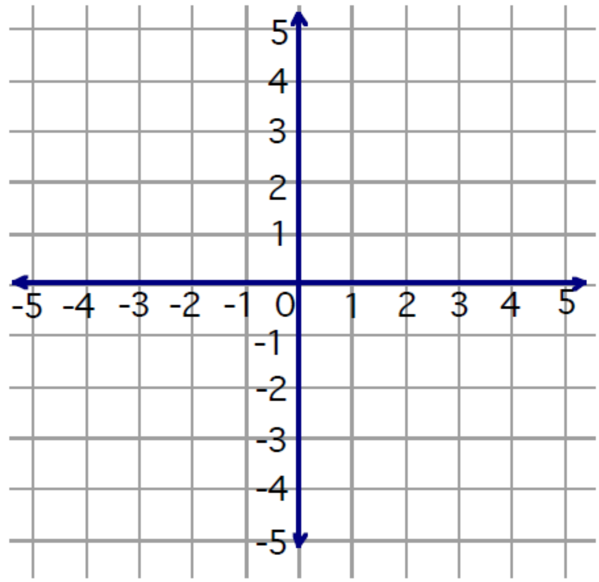
(A) 0.019 $\quad$ (B) 0.097 $\quad$ (C) 0.113 $\quad$ (D) 0.399 $\quad$ (E) 0.417

10-2 Plane Curves and Parametric Equations

Parametric equations: equations in terms of a third variable (usually t or θ)

EX #1: Graph $x = t^2 - 4$ $y = \frac{t}{2}$, $-2 \le t \le 3$ and show direction.

t	x	y
-2		
-1		
0		
1		
2		
3		



$\frac{dx}{dt}$ is the change in x over time.

$\frac{dy}{dt}$ is the change in y over time.

$\frac{dy}{dx}$ is the change y over change in x. (slope)

EX#2: Given $x = t^2 - 4$ $y = \frac{t}{2}$ Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and $\frac{dy}{dx}$ at $t = -1$

EX #3: Given $x = 3t^2 - 7t$ $y = e^{4t}$ Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and $\frac{dy}{dx}$ at $t = 2$.

10-3 Parametric Equations

If a smooth curve c is given by the equation, $x = f(t)$ and $y = g(t)$, then the slope of c at (x, y) is

$$Slope = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad , \quad \frac{dx}{dt} \neq 0 \qquad \text{Also, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}}$$

$$speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \text{ or } \sqrt{(x'(t))^2 + (y'(t))^2}$$

EX #1: $x = \sqrt{t} \qquad y = \frac{1}{4}(t^2 - 4) \qquad t \geq 0$ Find slope, concavity, and speed at $(2, 3)$.

$***$**Arclength in parametric form:**

If a smooth curve c is given by $x = f(t)$ and $y = g(t)$ such that c does not intersect on the interval $a \leq t \leq b$ (except possibly at the endpoints) then the arclength of c over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{or} \quad s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Also can be used to find distance particle travelled along the curve.

$***$**Reminder: Arclength in function form:**

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad \text{(Notes 7-4)}$$

EX #2: Find arclength of $x = 2\sin t$ and $y = 2\cos t \quad [0, 2\pi]$

Using Parametric Equations to solve Rectilinear Motion problems

EX #1: 1997 #1 $t = 0$ to $t = 6$ $x(t) = 3\cos(\pi t)$ $y(t) = 5\sin(\pi t)$

$a)$ Find the position of the particle when $t = 2.5$.

$b)$ Sketch from $t = 0$ to $t = 6$.

$c)$ The # of times it passes thru (0, 5)

$d)$ Find velocity vector

$e)$ Distance travelled from $t = 1.25$ to $t = 1.75$.

EX #2: 2000 #4 Position at $t = 1$ is $(2, 6)$ Velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$

$a)$ Find acceleration vector at $t = 3$.

$b)$ Find position at $t = 3$.

$c)$ When is slope = 8

$d)$ $\lim_{t \to \infty} \frac{dy}{dx} =$

EX #3: 2001 #1 $\frac{dx}{dt} = \cos(t^3)$, $\frac{dy}{dt} = 3\sin(t^2)$, $0 \le t \le 3$ At $t = 2$ position is $(4, 5)$.

$a)$ Equation of tangent line at $(4, 5)$

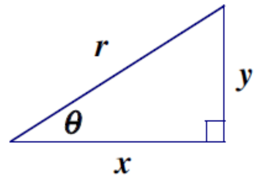
$b)$ Find speed at $t = 2$

$c)$ Distance travelled from $0 \le t \le 1$

$d)$ position at $t = 3$.

10-4 Polar Coordinates and Polar Graphs

Equations to convert rectangular to polar and polar to rectangular



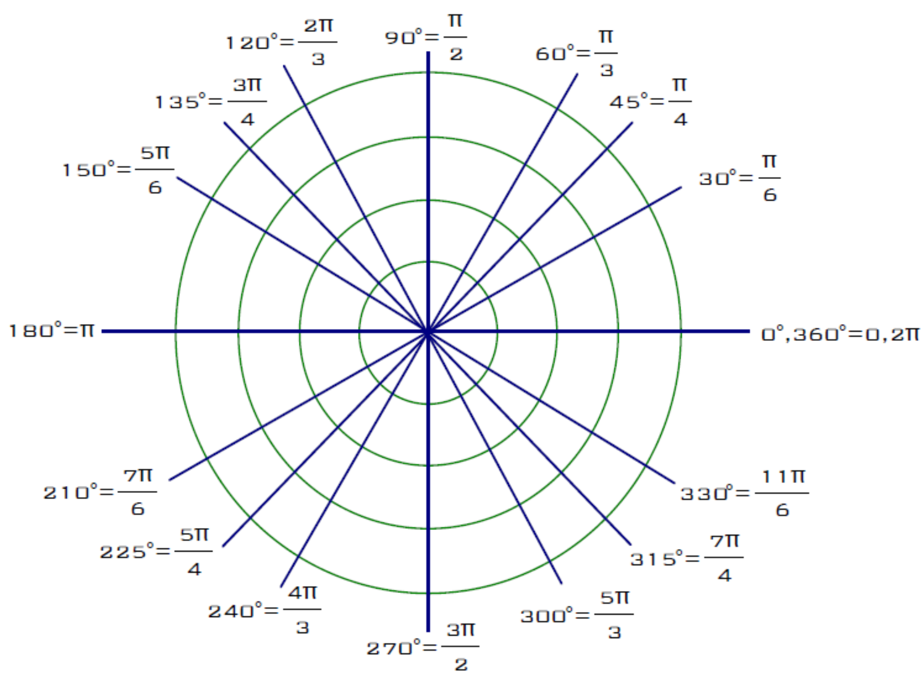
		polar to rectangular	rectangular to polar
Rectangular:	$(x,\ y)$	$x = r\cos\theta$	$x^2 + y^2 = r^2$
Polar:	$(r,\ \theta)$	$y = r\sin\theta$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

EX #1 : Convert each

$a)$ $(5, -12)$ $R \to P$

$b)$ $\left(-3, \frac{7\pi}{6}\right)$ $P \to R$

EX #2 : Graph $r = 1 + 2\cos\theta$



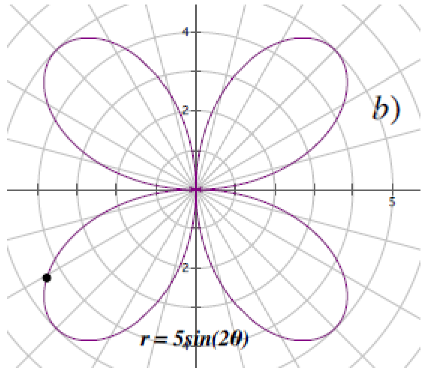
r	θ
	0
	$\pi/6$
	$\pi/4$
	$\pi/3$
	$\pi/2$
	$2\pi/3$
	$3\pi/4$
	$5\pi/6$
	π
	$7\pi/6$
	$5\pi/4$
	$4\pi/3$
	$3\pi/2$
	$5\pi/3$
	$7\pi/4$
	$11\pi/6$
	2π

Polar Equations

Remember : $\underline{x = r\cos\theta}$ and $\underline{y = r\sin\theta}$

EX#1: Given $r = 5\sin(2\theta)$.

$a)$ Find $\dfrac{dr}{d\theta}$ at $\theta = \dfrac{7\pi}{6}$.



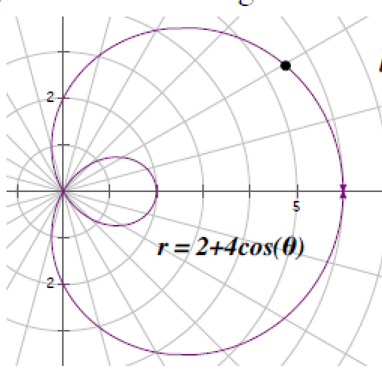
$b)$ Find $\dfrac{dx}{d\theta}$ at $\theta = \dfrac{7\pi}{6}$.

$c)$ Find $\dfrac{dy}{d\theta}$ at $\theta = \dfrac{7\pi}{6}$.

$d)$ Find $\dfrac{dy}{dx}$ at $\theta = \dfrac{7\pi}{6}$.

EX#2: Given $r = 2 + 4\cos\theta$, find the following:

$a)$ Find $\dfrac{dr}{d\theta}$ at $\theta = \dfrac{\pi}{6}$.



$b)$ Find $\dfrac{dx}{d\theta}$ at $\theta = \dfrac{\pi}{6}$.

$c)$ Find $\dfrac{dy}{d\theta}$ at $\theta = \dfrac{\pi}{6}$.

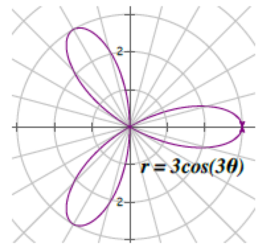
$d)$ Find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{6}$.

10-5 Area and Arclength in Polar Coordinates

<u>*Area in Polar Coordinates</u>**

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graphs of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\, d\theta$ $\quad\quad$ $***A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$



<u>EX #1:</u> $r = 3\cos 3\theta$ $\quad$ Find area of one leaf.

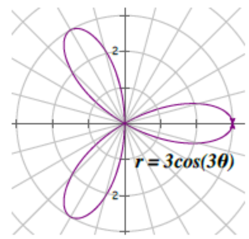
10-5 Area and Arclength in Polar Coordinates

*****Area in Polar Coordinates**

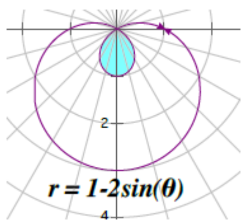
If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graphs of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\, d\theta \qquad \boxed{***A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta}$$

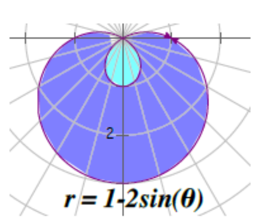
EX #1: $r = 3\cos 3\theta$ Find area of one leaf.



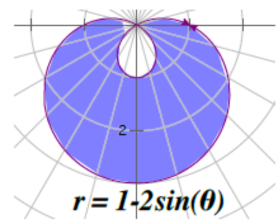
EX #2: $r = 1 - 2\sin\theta$



Area of the Inner Loop:



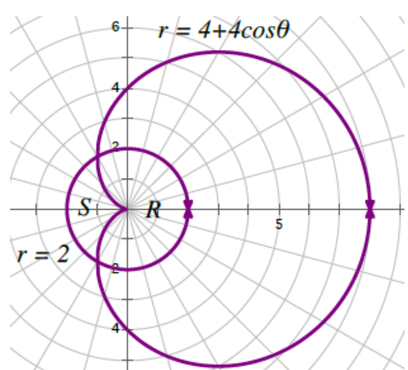
Area of the whole figure:



Area of the Outer Loop:

EX #3: $r = 4 + 4\cos\theta$ $r = 2$

$a)$ Find area of region R.



$b)$ Find area of region S.

Arclength of a polar curve (Not on AP Test): $s = \int_{\alpha}^{\beta}\sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\, d\theta$ or $s = \int_{a}^{b}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

<u>Hooke's Law</u>

<u>Hooke's Law:</u> The Force required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed from its original length.

$$F = kd$$

$$W = \int_a^b kx\,dx$$

<u>EX#1 :</u> A force of 750 lbs compresses a spring 3 inches from its natural length of 15 inches. Find the work done in compressing the spring an additional 3 inches.

<u>EX#2 :</u> (1997 MC) A force of 10 lbs is required to stretch a spring 4 inches beyond its natural length. Assuming Hooke's Law applies, how much work is done in stretching the spring from its natural length to 6 inches beyond its natural length.

Finding Area Using Limits

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(a+\frac{b-a}{n}i\right)}_{height}\underbrace{\left(\frac{b-a}{n}\right)}_{width} \qquad i = \text{interval}, \quad n = \text{\# of subdivisions}$$

$$Area = \left(\frac{b-a}{n}\right)\left[f\left(a+\frac{b-a}{n}\cdot 1\right)+f\left(a+\frac{b-a}{n}\cdot 2\right)+f\left(a+\frac{b-a}{n}\cdot 3\right)+\text{.........}f\left(a+\frac{b-a}{n}\cdot n\right)\right]$$

Summation Formulas

1) $\sum_{i=1}^{n} c = cn$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Summation Properties

5) $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

6) $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, where k is a constant

EX#1: $f(x) = x^3 \quad [0,1] \quad n$ subdivisions

EX#2: $f(x) = x^2 \quad [1,3] \quad n$ subdivisions

EX#3: 1988 MC #41 $\quad \lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\text{.....}\sqrt{\frac{n}{n}}\right] =$

A) $\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}dx$ B) $\int_0^1 \sqrt{x}\,dx$ C) $\int_0^1 x\,dx$ D) $\int_1^2 x\,dx$ E) $2\int_0^1 x\sqrt{x}\,dx$

EX#4: $\lim_{n\to\infty} \frac{2}{n}\left[\left(3+\frac{2}{n}\right)^5+\left(3+\frac{4}{n}\right)^5+\left(3+\frac{6}{n}\right)^5+\text{......}+\left(3+\frac{n}{n}\right)^5\right] =$