§11.1—Sequences & Series: Convergence & Divergence

A sequence is simply list of things generated by a rule

More formally, a **sequence** is a function whose domain is the set of positive integers, or **natural numbers**, n, such that $n \in \mathbb{N} = \{1,2,3,\ldots\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Where a_n is called the ***n*th term** (or rule of sequence), and we denote the sequence by $\{a_n\}$.

The sequence can be expressed by either
1) an ample number of terms in the sequence, separated by commas
2) an explicit function defined by the **rule of sequence**
3) the rule of sequence set off in braces.

Example 1:
The sequence $2,4,6,8,\ldots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer n. The sequence $1,3,5,\ldots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer n. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

'RULE OF 3'

BLOOD, SWEAT, TEARS
LIFE, LIBERTY, + PURSUIT OF HAPPINESS
GOVERNMENT OF PEOPLE, BY PEOPLE, FOR PEOPLE
FATHER, SON, + HOLY SPIRIT
STOP, LOOK, LISTEN

***NOTE: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using n as an ordinal (or counting) number, rather than a cardinal number.

We will be primarily interested in what happens to the sequence for increasingly large values of n.

Example 2:
If $a_n = \left\{\dfrac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim\limits_{n\to\infty} a_n$.

$\frac{8}{7}$ $\frac{12}{9} \to \frac{4}{3}$ $\frac{16}{11}$ $\frac{20}{13}$

$\left\{\frac{4}{3}, \frac{8}{7}, \frac{4}{3}, \frac{16}{11}, \frac{20}{13}\right\}$ $\lim\limits_{n\to\infty} a_n \approx 2$

FACT:

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) If $\lim_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Definition:

$n!$ is read as "*n* factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

$9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

Example 3:
Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

CONVERGES; $\to 1$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

CONVERGES; $\to 0$

(c) $a_n = 3 + (-1)^n$

DIVERGES; OSCILLATES BETWEEN 2 + 4

(d) $a_n = \frac{n}{1-2n}$

CONVERGES; $\to -\frac{1}{2}$

(e) $a_n = \frac{\ln n}{n}$

CONVERGES; $\to 0$

(f) $a_n = \frac{n!}{(n+2)!}$

CONVERGES; $\to 0$

(g) $a_n = \frac{2n!}{(n-1)!}$

(h) $a_n = \frac{n+(-1)^n}{n}$

DIVERGES; OSCILLATES

(i) $a_n = \frac{(-1)^n (n-1)}{n}$

CONVERGES; $\to 1$
ALTERNATING SERIES

(j) $a_n = \frac{2^n}{(n+1)!}$

CONVERGES; $\to 0$

(k) $a_n = \left(1 + \frac{1}{n}\right)^n$

CONVERGES; $\to e$

(l) $\left\{ \frac{(2n)!}{n^n} \right\}$

DIVERGES; $\to \infty$

Sometimes, albeit rarely, we have to write the rule of sequence as a function of n from a pattern.

Example 4:
Write an expression for the nth term.

(a) 3, 8, 13, 18, . . . ARITHMETIC

$a_n = 5n - 2$

$3+5(n-1)$ $a+(n-1)d$
$3+5n-5$

(b) 5, -15, 45, -135, . . . GEOM.

$a_n = 5(-3)^{n-1}$

ar^{n-1} $\cancel{5(-3)^{n-1}}$

$\cancel{5+(-3)(n-1)}$ $\cancel{5+3n+3}$ $2+3n$

(c) 1, 4, 9, 16, 25, . . . GEOM

$a_n = n^2$

(d) 4, 10, 28, 82, . . .

$a_n = 3^n + 1$

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

$a_n = \frac{n+1}{2n-1}$

(f) $\ln 1,\ \ln 2,\ \ln 4,\ \ln 8, \ldots$

$a_n = \ln(2^{n-1})$