11.2 Series

While much more can be said about sequences, we now turn to our principal interest, series. Recall that a series, roughly speaking, is the sum of a sequence: if $\{a_n\}_{n=0}^{\infty}$ is a sequence then the associated series is

$$\sum_{i=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Associated with a series is a second sequence, called the **sequence of partial sums** $\{s_n\}_{n=0}^{\infty}$:

$$s_n = \sum_{i=0}^{n} a_i.$$

So

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \quad \ldots$$

A series converges if the sequence of partial sums converges, and otherwise the series diverges.

EXAMPLE 11.2.1 If $a_n = kx^n$, $\sum_{n=0}^{\infty} a_n$ is called a **geometric series**. A typical partial sum is

$$s_n = k + kx + kx^2 + kx^3 + \cdots + kx^n = k(1 + x + x^2 + x^3 + \cdots + x^n).$$

We note that

$$\begin{aligned}s_n(1-x) &= k(1+x+x^2+x^3+\cdots+x^n)(1-x)\\ &= k(1+x+x^2+x^3+\cdots+x^n)1 - k(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)x\\ &= k(1+x+x^2+x^3+\cdots+x^n - x - x^2 - x^3 - \cdots - x^n - x^{n+1})\\ &= k(1-x^{n+1})\end{aligned}$$

so

$$\begin{aligned}s_n(1-x) &= k(1-x^{n+1})\\ s_n &= k\frac{1-x^{n+1}}{1-x}.\end{aligned}$$

If $|x|<1$, $\lim_{n\to\infty} x^n = 0$ so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} k\frac{1-x^{n+1}}{1-x} = k\frac{1}{1-x}.$$

Thus, when $|x|<1$ the geometric series converges to $k/(1-x)$. When, for example, $k=1$ and $x=1/2$:

$$s_n = \frac{1-(1/2)^{n+1}}{1-1/2} = \frac{2^{n+1}-1}{2^n} = 2 - \frac{1}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2.$$

We began the chapter with the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n},$$

namely, the geometric series without the first term 1. Each partial sum of this series is 1 less than the corresponding partial sum for the geometric series, so of course the limit is also one less than the value of the geometric series, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

□

It is not hard to see that the following theorem follows from theorem 11.1.2.

THEOREM 11.2.2 Suppose that $\sum a_n$ and $\sum b_n$ are convergent series, and c is a constant. Then

1. $\sum ca_n$ is convergent and $\sum ca_n = c\sum a_n$

2. $\sum(a_n + b_n)$ is convergent and $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

■

The converses of the two parts of this theorem are subtly different. Suppose that $\sum a_n$ diverges; does $\sum ca_n$ also diverge if c is non-zero? Yes: suppose instead that $\sum ca_n$ converges; then by the theorem, $\sum(1/c)ca_n$ converges, but this is the same as $\sum a_n$, which by assumption diverges. Hence $\sum ca_n$ also diverges. Note that we are applying the theorem with a_n replaced by ca_n and c replaced by $(1/c)$.

Now suppose that $\sum a_n$ and $\sum b_n$ diverge; does $\sum(a_n + b_n)$ also diverge? Now the answer is no: Let $a_n = 1$ and $b_n = -1$, so certainly $\sum a_n$ and $\sum b_n$ diverge. But $\sum(a_n + b_n) = \sum(1 + -1) = \sum 0 = 0$. Of course, sometimes $\sum(a_n + b_n)$ will also diverge, for example, if $a_n = b_n = 1$, then $\sum(a_n + b_n) = \sum(1 + 1) = \sum 2$ diverges.

In general, the sequence of partial sums s_n is harder to understand and analyze than the sequence of terms a_n, and it is difficult to determine whether series converge and if so to what. Sometimes things are relatively simple, starting with the following.

THEOREM 11.2.3 If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} s_{n-1} = L$, because this really says the same thing but "renumbers" the terms. By theorem 11.1.2,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

But

$$s_n - s_{n-1} = (a_0 + a_1 + a_2 + \cdots + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_n,$$

so as desired $\lim_{n\to\infty} a_n = 0$. ■

This theorem presents an easy divergence test: if given a series $\sum a_n$ the limit $\lim_{n\to\infty} a_n$ does not exist or has a value other than zero, the series diverges. Note well that the converse is *not* true: If $\lim_{n\to\infty} a_n = 0$ then the series does not necessarily converge.

EXAMPLE 11.2.4 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

We compute the limit:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots + 1 + 1 + 1 + 1 + \cdots$, and of course if we add up enough 1's we can make the sum as large as we desire. □

EXAMPLE 11.2.5 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Here the theorem does not apply: $\lim_{n\to\infty} 1/n = 0$, so it looks like perhaps the series converges. Indeed, if you have the fortitude (or the software) to add up the first 1000 terms you will find that

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.49,$$

so it might be reasonable to speculate that the series converges to something in the neighborhood of 10. But in fact the partial sums do go to infinity; they just get big very, very slowly. Consider the following:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+\frac{1}{2}+\frac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{16} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{16} = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another $1/2$ to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} > 1+\frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4 \cdot 10^{59}$ terms. This series, $\sum(1/n)$, is called the **harmonic series**. □

Exercises 11.2.

1. Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ diverges. ⇒
2. Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{1/n}+14}$ diverges. ⇒
3. Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n}$ diverges. ⇒

4. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - \frac{3}{3^n}$. $\Rightarrow$

5. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n}$. $\Rightarrow$

6. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$. $\Rightarrow$

7. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}}$. $\Rightarrow$

8. Compute $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$. $\Rightarrow$

9. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$. $\Rightarrow$