BC Calculus

Sec 9.2
Taylor & Maclaurin
Series

Find $\frac{d}{dx}\sum_{n=0}^{\infty} 4(2x)^n$

$$\frac{d}{dx}\sum_{n=0}^{\infty} 4(2x)^n = \sum_{n=1}^{\infty} 4n(2x)^{n-1}(2)$$

Chain rule

$$= \sum_{n=1}^{\infty} 8n(2x)^{n-1}$$

Find $\int \sum_{n=0}^{\infty} 5x^n dx$

$$\int \sum_{n=0}^{\infty} 5x^n dx = c + \sum_{n=0}^{\infty} \frac{5x^{n+1}}{n+1}$$

Find $\int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Find the power series representation for f(x)= $\tan^{-1} x$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\left(\tan^{-1} x\right) = \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int \sum_{n=0}^{\infty}(-1)^n x^{2n} dx = c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

The *n* factor

Power Series' Interval of convergence

Interpreting the Ratio Test

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)(3x-5) = (3x-5) \quad \left(\frac{n+1}{n} \to 1\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}\right)(3x-5) = 0 \quad \left(\frac{1}{n+1} \to 0\right)$$

$$\lim_{n\to\infty}(n+1)(3x-5) = \begin{cases} \pm\infty & 3x-5 \neq 0 \\ 0 & 3x-5 = 0 \end{cases} \quad \left((n+1) \to \infty\right)$$

The *n* factor

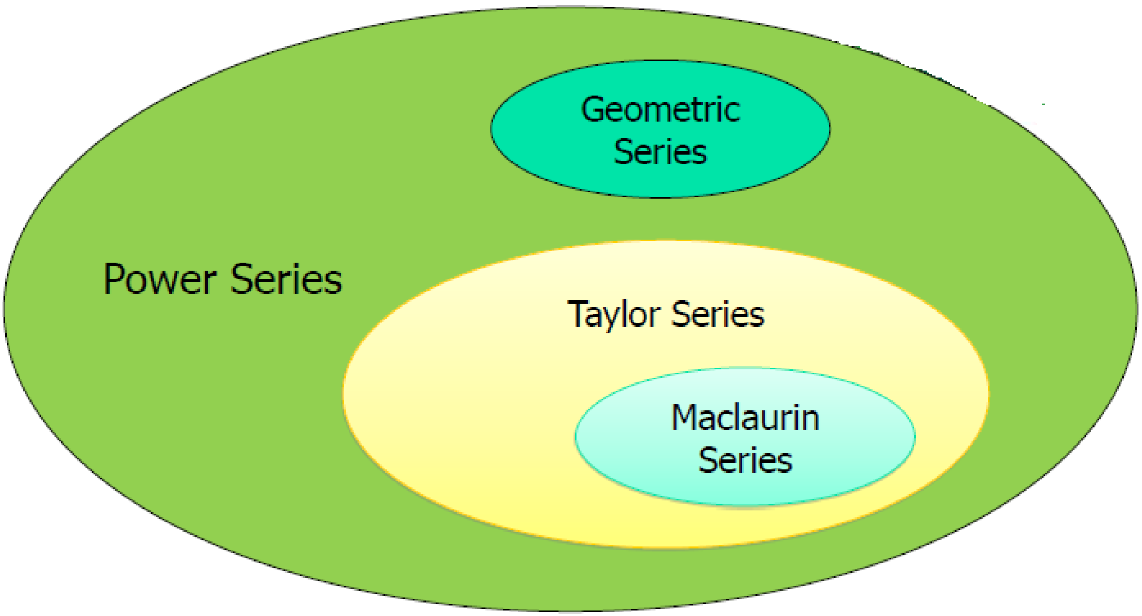
Power Series' Interval of convergence
Interpreting the Ratio Test

$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)(3x-5)=(3x-5)$ Converges when $-1<(3x-5)<1$

$\lim_{n\to\infty}\left(\frac{1}{n+1}\right)(3x-5)=0$ It converges for all real #s

$\lim_{n\to\infty}(n+1)(3x-5)=\begin{cases}\pm\infty & 3x-5\neq 0\\ 0 & 3x-5=0\end{cases}$ It diverges EXCEPT when $3x-5=0$

Our story so far



Why are we bothering?

$$\cos 0 = 1 \qquad \cos 2 = ?$$

$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{2} = ?$$

$$\ln 1 = 0 \qquad \ln 2 = ?$$

$$e^1 = e \qquad e^2 = ?$$

Easily memorized and recalled

Series give us the means to evaluate functions like these

MEMORIZE...

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Converges for all real #s

What is the interval of convergence?

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| \quad = \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \quad = \frac{x^2}{(2n+1)(2n+2)} \quad = 0$$

$0 < 1$ Regardless of what x equals

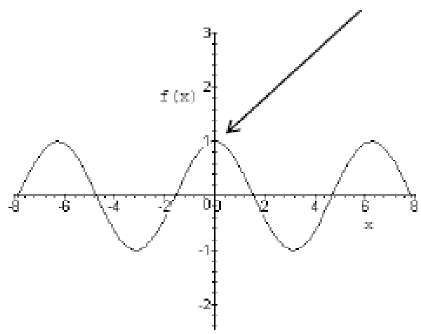
$\therefore$ Series converges for all real #s

Think about it

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$ For all real #s

And all we needed to know was

how $\cos x$ behaves at $x = 0$.



Now, let's find Sin

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

integrate ↓ integrate ↓

$$\sin x = c + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Let $x = 0$

$\sin 0 = c$

$c = 0$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Interval of Convergence

Integration does not change the interval of convergence.

So the Sin series also converges for all real #s.

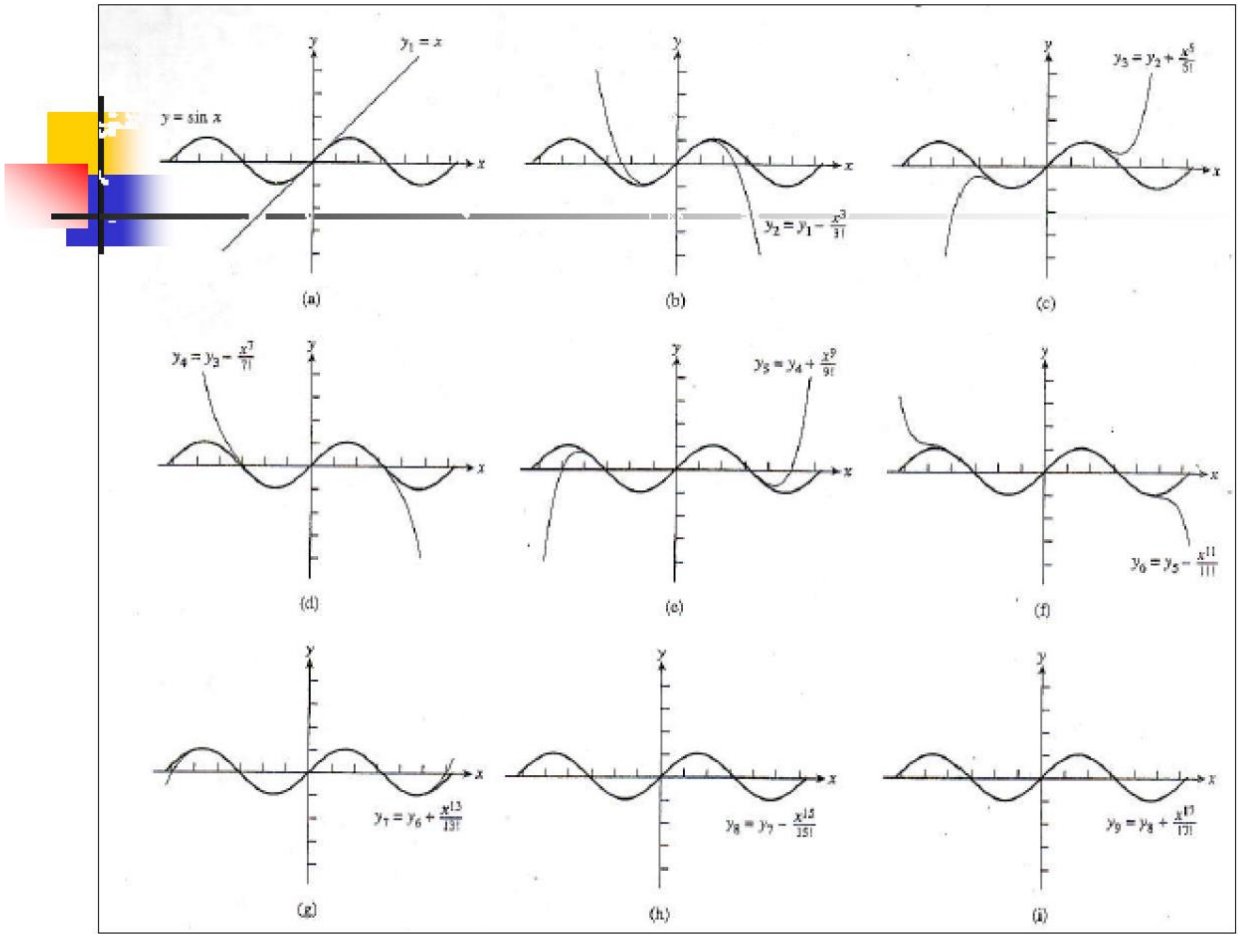
MEMORIZE

Summary

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

For all real #s



y = sin x
$y_1 = x$
$y_2 = y_1 - \frac{x^3}{3!}$
$y_3 = y_2 + \frac{x^5}{5!}$
$y_4 = y_3 - \frac{x^7}{7!}$
$y_5 = y_4 + \frac{x^9}{9!}$
$y_6 = y_5 - \frac{x^{11}}{11!}$
$y_7 = y_6 + \frac{x^{13}}{13!}$
$y_8 = y_7 - \frac{x^{15}}{15!}$
$y_9 = y_8 + \frac{x^{17}}{17!}$
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)

Now, let's generalize

Original Function — 1^{st} derivative — 2^{nd} derivative — 3^{rd} derivative

$$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \cdots + c_n x^n + \cdots$$

You can see the factorial and exponent are the same.

And, they are the same as the derivative #.

So, what is the nth term?

$$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \cdots + c_n x^n + \cdots$$

$$n^{th} \text{ term} = \frac{f^n(0)}{n!}x^n$$

Example:

$$20^{th} \text{ term} = \frac{f^{20}(0)}{20!}x^{20}$$

Generalizing

If series is centered at x = a,

The nth derivative at x = a

$$n^{th}\ \text{term} = \frac{f^{n}(a)}{n!}(x-a)^{n}$$

n factorial

(x – a) to the nth power

Definition: Taylor Series

if *f* is a function with derivatives of all orders throughout some open interval containing *a*, then:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^{2} + \frac{f'''(a)}{3!}(x-a)^{3} + \ldots$$

0! = 1, so the first term will always end up being f(a).

A Taylor Series centered at ***a*** **= 0** is known as a **Maclaurin Series**.

Ex) Find the Taylor series for f(x) = e^x and its interval of convergence at x = 0

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$$

n	Derivative	At $x = 0$
0	$f(x)=e^x$	$f(0)=1$
1	$f'(x)=e^x$	$f'(0)=1$
2	$f''(x)=e^x$	$f''(0)=1$
3	$f'''(x)=e^x$	$f'''(0)=1$
4	$f^4(x)=e^x$	$f^4(0)=1$

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\dots$$

Let's build our series

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Example: $e^5 = 1+5+\frac{5^2}{2}+\frac{5^3}{3!}+\cdots$

To find the interval of convergence, do Ratio Test on $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{x}{n+1} \to 0 < 1$$

The series converges for all real #s and the radius of convergence is R = ∞

And we only needed to know behavior at $x = 0$

Memorize These

Taylor Series $= \sum \frac{f^n(a)}{n!}(x-a)^n$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad (\text{all real \#s})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)\,!} + \cdots \qquad (\text{all real \#s})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)\,!} + \cdots \qquad (\text{all real \#s})$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad (-1 < x < 1)$$

Another Ex) Find a power series expansion for ln(x) centered at x = 1.

n	Derivative	At $x = 1$
0	$f(x) = \ln x$	$f(1) = 0$
1	$f'(x) = \frac{1}{x}$	$f'(1) = 1$
2	$f''(x) = -\frac{1}{x^2}$	$f''(1) = -1$
3	$f'''(x) = \frac{2}{x^3}$	$f'''(1) = 2!$
4	$f^4(x) = \frac{-2 \cdot 3}{x^4}$	$f^4(1) = -2 \cdot 3 = -3!$
5	$f^5(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$	$f^5(1) = 2 \cdot 3 \cdot 4 = 4!$

When finding the derivatives, look for the factorial patterns.

Now let's write our series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$f(x)=0+1(x-1)-\frac{(x-1)^{2}}{2!}+\frac{2!(x-1)^{3}}{3!}-\frac{3!(x-1)^{4}}{4!}+\ldots$$

$$f(x)=(x-1)-\frac{(x-1)^{2}}{2}+\frac{(x-1)^{3}}{3}-\frac{(x-1)^{4}}{4}+\cdots+(-1)^{n-1}\frac{(x-1)^{n}}{n}+\cdots$$

Find the Interval of Convergence

$$\ln(x-1) = \sum (-1)^{n-1} \frac{(x-1)^n}{n}$$

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right| = \lim_{n\to\infty} \left| (x-1) \cdot \frac{n}{n+1} \right| = (x-1)$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

Series techniques still apply

Find the Maclaurin series for $\dfrac{(1+\cos 2x)}{2}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \text{(from earlier)}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$1 + \cos 2x = 2 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$\frac{1+\cos 2x}{2} = \frac{2}{2} - \frac{(2x)^2}{2!2} + \frac{(2x)^4}{4!2} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!2} + \cdots$$

$$\frac{1+\cos 2x}{2} = 1 - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \cdots + (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad g(x) = \frac{e^x - 1}{x^2}$$

Find the 1st three terms of a series for g(x) and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x - 1 = \quad x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\frac{e^x - 1}{x^2} = \frac{x}{x^2} + \frac{x^2}{x^2 2!} + \frac{x^3}{x^2 3!} + \cdots + \frac{x^n}{x^2 n!} + \cdots$$

$$= x^{-1} + \frac{1}{2!} + \frac{x}{3!} + \cdots \quad + \frac{x^{n-2}}{n!} + \cdots$$