Integral Test and p-Series

Integral Test

If f is **D**ecreasing, **C**ontinuous, and **P**ositive (**D**ogs **C**uss in **P**rison!) for $x \geq 1$ AND $a_n = f(x)$,

then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either BOTH converge or diverge.

Note 1: This does NOT mean that the series converges to the value of the definite integral!!!!!!

Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$.

If the series converges to S, then the remainder, $R_n = |S - S_n|$ is bounded by

$0 \leq R_n \leq \int_n^{\infty} f(x)dx$. (Not on AP exam, but on my exam.). This means $S \in [S_n, S_n + R_n]$.

Example 11:

Determine whether the following series converge or diverge. If they converge, find an interval in which the sum resides using S_4.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$f(x) = \frac{x}{x^2+1}$ IS POS. + CONT FOR $x \geq 1$

$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$, SO $f'(x) < 0$ FOR $x > 1$

$t = x^2+1$, $dt = 2x\,dx$ $\Rightarrow \lim_{n\to\infty} \int_2^{x^2+1} \frac{dt/2}{t} \Rightarrow \frac{1}{2}\lim_{n\to\infty} \int_2^{x^2+1} \frac{1}{t}dt = \frac{1}{2}\lim_{n\to\infty} \ln t \Big|_2^{x^2+1}$

$\Rightarrow \frac{1}{2}\lim_{n\to\infty} [\ln(x^2+1) - \ln 2] = \infty$ SINCE GIVEN INTEGRAL DIV THE INITIAL SERIES ALSO DIV.

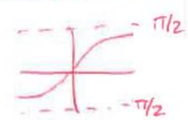
(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$ IS POS + CONT FOR $x \geq 1$

$f'(x) = \frac{-2x}{(x^2+1)^2}$, SO $f'(x) < 0$ FOR $x > 1$

$\lim_{n\to\infty} \int_1^n \frac{1}{n^2+1} dx$ IMPROPER INT

$\int_1^{\infty} \frac{dx}{x^2+1} \Rightarrow \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan b - \arctan 1$



$\Rightarrow \frac{\pi}{2} - \frac{\pi}{4}$ $\therefore$ THE SERIES CONVERGES

Example 12:

CONVERGES

Approximate the sum of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using six terms. Include an estimate of the maximum error for your approximation.

$f(x) = \frac{1}{x^4}$ POS+ CONT $x \geq 1$

SUM(SEQ($\frac{1}{n^4}$, n, 1, 500)) ≈ 1.082

THIS IS A p-SERIES W/ $p = 4$, SO IT CONVERGES

$f'(x) = -4x^{-5} < 0$ $x \geq 1$

$$\int_n^{\infty} \frac{1}{x^4} dx = \lim_{t \to \infty} \int_n^t \frac{1}{x^4} dx = \lim_{t \to \infty} \left[\frac{-1}{3x^3} \right]_n^t = \frac{1}{3n^3}$$

FOR ERROR